DISSOLUTION OF TRANSFORMATION PROBLEM IN THREE-DEPARTMENTS MODEL OF SIMPLE PRODUCTION[1].

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ABSTRACT[2].

The dissolution of transformation problem in three-departments model of simple production is proposed. Four models are considered in this paper. Models-1 and 2 don't take into account the "labor of capitalists" (as managers and entrepreneurs). Models-3 and 4 take into account this factor. Embodied labor (value) in models-1 and 2 consists of labor of workers. Embodied labor (value) in models-3 and 4 consists of labor of both workers and capitalists. Capitalists are expending net profit (= gross profit - salary of capitalists as entrepreneurs) on "luxury goods" in models-1 and 3 and on "luxury goods" and "subsistence goods" in models-2 and 4. The model-4 is the most realistic model of capitalist simple production. The solution of transformation problem exists in the models-1 and 2 if non-trivial balance-conditions are superimposed onto the economy. Marx has introduced non-trivial balance-conditions for the simple production in XX chapter of second volume of "Capital". We argue that these conditions were carried out in early capitalist economy (merchant capitalism). The problem statement in Bortkiewicz' (1907) paper (in frame of model-1) doesn't take into account Marx's non-trivial balance-conditions. The process of historical transformation is considered theoretically and it is modeled numerically. The solution of transformation problem in frame of model-4 is proposed. Solution in the model-4 exists without superimposition of Marx's non-trivial balance-conditions.

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[2] It is the expanded version of paper published earlier: http://www.socintegrum.ru/P_2011_1_en.html

I. Introduction.

Bortkiewicz (1907a) in 1907 has considered the transformation of “values” into “prices of production” in three-departments model of simple production (“means of production” (I); “consumer goods” (II) and “luxury goods” (III)). He found that Marx’s transformational rules couldn’t be executed simultaneously in the economy with arbitrary structure. Only one rule of transformation can be executed in the general case: either the sum of profits equals to aggregate surplus value or price and value of aggregate output are equal. Bortkiewicz concluded about logical inconsistence in Marx’s theory. So there was the “transformation problem”.

This discouraging result shattered confidence in Marx’s “labor theory of value”. Later many scientists came to the same conclusion (Sweezy (1949); Meek (1956); Samuelson (1957, 1971); Medio (1972); Steedman, I. (1977); Abraham-Frois, G. (1979); Itoh (1980)) and now after a century majority of economists is convinced that Marx's theory is internally inconsistent.

Numerous attempts have been undertaken in order to dissolve this problem. Morishima and Caterhores (1975) have assumed that Marxian algorithm of transformation is only the first step of iterative process. Shaikh (1977; 1984) has assumed that the solution of this problem can be obtained as a result of many iterations of the Marx’s transformation algorithm. Sweezy (1949) has assumed that the problem can be dissolved if in reality “values” and “prices of production” are connected by nonlinear relation. Lipietz (1982), Dumenil (1980; 1983), and Foley (1982) have offered new interpretation of “transformational rules” – so-called “new solution”. Freeman (1996), Kliman and McGlone (1999) and Kliman (2007) have offered so-called "temporal single-system interpretation" (TSSI) in which "output prices" and "input prices" can differ in each period because the real economy is non-equilibrium dynamical system.

Despite elegance of many ideas offered for dissolution of this problem, among Marxists-theorists till now there is not any consent concerning the given problem. Moseley (1999), Fine et al. (2004), Mariolis (2006) and a number of other authors criticize "new solution", noticing that it differs from initial statement of this problem and comprises the implicit tautology. Laibman (2004), Roberto (2004), Park (2009) recently have put forward serious objections against TSSI-approach to a solution of transformation problem. So, the transformation problem still remains the problem which does not have any conventional solution.

We reconsider once again Bortkiewicz’ model. Was the problem stated correctly? The model of simple reproduction assumes the performance of certain balance conditions in economy. We argue in this paper that Bortkiewicz’ problem statement doesn’t take into account all necessary conditions of balanced exchange for the early capitalist economy. As consequence the set of solutions in Bortkiewicz’ paper is wider than it was possible in real early capitalist economy with simple reproduction. Superfluous solutions obtained by Bortkiewicz corresponds to non-equilibrium structures of early capitalist economy and don't satisfy to all Marx’s rules of transformation. If solutions correspond to all necessary balance conditions for the early capitalist economy then all Marxian transformational rules are carried out. Equilibrium structure of early capitalist economy satisfied to some non-trivial balance-conditions. Marx is introduced these balance-conditions in chapter XX of second volume of “Capital”.

Paper is structured as follows. Section II contains general description of three-departments models of simple production. Section III is devoted to logical analysis of “value-composition” of economy. We introduce “value-matrix” and “matrix of input-flows” for different types of exchange. Sections IV-V are devoted to consideration of the model-1 for three partial cases: 1) exchange on the base of "values"; 2) exchange on the base of "prices of production"; and 3) exchange at which "values" and "prices of production" coincide. Each mode of exchange corresponds to the definite "value-structure". We prove that Marxian transformational rules are carried out in the economy with non-trivial balance-conditions if equilibrium prices are equal to "prices of production". We discuss in details in Section VI why the economy with simple production must satisfy to these non-trivial balance-conditions. We demonstrate that non-trivial balance-conditions follow from Marx's analysis of simple reproduction in XX chapter of the second volume of "Capital". Section VII is devoted to analysis of Bortkiewicz' (1907a) problem statement. Section VIII discusses the possible variants of "historical transformation". We consider two modes of "historical transformation" and construct the numerical model for each case. Section IX contains the dissolution of transformation problem in model-2 in which capitalists consume both "consumer goods" and "luxury goods". Section X contains solution of transformation problem in frame of model-4 in which the labor of capitalists is taken into account. Section XI is devoted to description of algorithms by means of which model-4 can be transformed into the model-2 and model-2 can be transformed into the model-1. Supplements to this paper contain the numerical examples, results of modeling, and Excel-file with program of calculations.

II. Three-departments Model of Simple Production.

We consider four models of simple production. We assume in all models that workers buy and consume only "necessities of life" whereas capitalists can buy and consume both "necessities of live" and "luxury goods".

Models-1 and 2 don't take into account the labor of capitalists as employees (managers and directors at their own enterprises). Labor of capitalists doesn't enter into value of product as some part of embodied socially necessary labor. Profits don't contain any compensation of capitalists as employees (managers and directors at their own enterprises).

Model-4 (and model-3) takes into account the labor of capitalists. Labor of capitalists enters into value of product as some part of embodied socially necessary labor. Part of profit (in economic sense) is compensation of capitalists as employees (managers and directors at their own enterprises). We have the following Classification Table of models.

Classification Table of models of capitalist economies with simple production[3].

MODEL	Labor of capitalists enters into value of product	Part of profit is compensation capitalists as employees	Compensation ("wage of capitalists" as employees) is expended upon:	Net profit (profit minus "wage of capitalists") is expended on:	Workers buy:
Model-1	No	No	No compensation	"luxury goods"	"necessities of life"
Model-2	No	No	No compensation	"luxury goods" and "necessities of life"	"necessities of life"
Model-3	Yes	Yes	"necessities of life"	"luxury goods"	"necessities of life"
Model-4	Yes	Yes	"necessities of life"	"luxury goods" and "necessities of life"	"necessities of life"

Bortkiewicz (1907) considered model-1. Marx in the second volume of "Capital" (chapter XX) considered the Model-2. We will consider in this paper the model-1, model-2 and model-4.

[3] The complete list of models is presented in **Supplement I.**

Model-4 is the most realistic model of capitalist economy. Capitalists are proprietors but they are also managers and directors at their own enterprises. They work as well as employees at their enterprises. Profit contains "wage" of capitalists. Marx emphasized many times this point:

"…the process of production, separated from capital, is simply a labour-process. Therefore, the industrial capitalist, as distinct from the owner of capital, does not appear as operating capital, but rather as a functionary irrespective of capital, or, as a simple agent of the labour-process in general, as a labourer, and indeed as a wage-labourer…

…The specific functions which the capitalist as such has to perform, and which fall to him as distinct from and opposed to the labourer, are presented as mere functions of labour. He creates surplus-value not because he works as a capitalist, but because he also *works, regardless of his capacity of capitalist. This portion of surplus-value is thus no longer surplus-value, but its opposite, an equivalent for labour performed…*

The conception of profit of enterprise as the wages of supervising labour, arising from the antithesis of profit of enterprise to interest, is further strengthened by the fact that a portion of profit may, indeed, be separated, and is separated in reality, as wages, or rather the reverse, that a portion of wages appears under capitalist production as integral part of profit…" (K. Marx "Capital", v. III, ch. XXIII)

"…[Consequently] the industrial capitalist as distinct from himself as capitalist, that is, the industrialist in contradistinction to himself as capitalist, i.e., owner of capital, is thus merely a simple functionary in the labour process; he does not represent functioning capital, but is a functionary irrespective of capital, and therefore a particular representative of the labour process in general, a worker. In this way, industrial profit is happily converted into wages *and is equated with ordinary wages, differing from them only quantitatively and in the special form in which they are paid, i.e., that the capitalist pays wages to himself instead of someone else paying them to him…*

…Therefore, insofar as the capitalist plays any part in it, he does so not as a capitalist—for this aspect of his character is allowed for in interest—but as a functionary of the labour process in general, as a worker, and his wages take the form of industrial profit. It is a special type of labour—labour—of superintendence—but after all types of labour in general differ from one another.

Industrial profit does indeed include some part of wages—in those cases where the manager does not draw them. Capital appears in the production process as the director of labour, as its commander (captain of industry) and thus plays an active role in the labour process… This work (it may be entrusted to a manager) which is linked with exploitation is, of course, labour which, in the same way as that of the wage-worker, enters into the value of the product…" (K. Marx, "Capital", v. IV ("Theories of surplus-values"), part. III, Addenda 4).

Capitalists in real life buy both "consumer goods" and "luxury goods" and the profit of capitalists consists of "net profit" and "wage" of capitalists as entrepreneurs ("profit" = "wage of capitalists" + "net profit"). Consequently models-1 and 3 are simplified (crude) models whereas the model-4 is more realistic model of real capitalist economy in frame of three-departments model.

Let's introduce two matrices: 1) matrix of values and 2) matrix of input-flows. The first matrix describes values of commodities which are consumed by each department during the production period. Workers in each department buy "necessities of life", capitalists buy "means of production" (equipment, raw materials, fuel, semi-finished products), "necessities of life" and "luxury goods". The second matrix contains these articles of expenses in prices of balanced exchange.

Table 1. Matrix-I of values.

Dept.	Value of "means of production". (1)	Value of "consumer goods". (2)	Value of "luxury goods". (3)	Sum of values consumed in department ("labor commanded"). (4)	Value of output ("labor cost"). (5)
I.	C_1	V_1	M_1	$C_1+V_1+M_1$	C
II.	C_2	V_2	M_2	$C_2+V_2+M_2$	V
III.	C_3	V_3	M_3	$C_3+V_3+M_3$	M
Σ	$C=\sum_{n=1}^{3} C_n$	$V=\sum_{n=1}^{3} V_n$	$M=\sum_{n=1}^{3} M_n$	$C+V+M$	$C+V+M$

Table 2. Matrix-II of input-flows.

Dept.	Price of "means of production". (1)	Price of "consumer goods". (2)	Price of "luxury goods". (3)	Sum of prices of goods consumed in department. (4)	Price of output. (5)
I.	xC_1	yV_1	zM_1	$xC_1+yV_1+zM_1$	xC
II.	xC_2	yV_2	zM_2	$xC_2+yV_2+zM_2$	yV
III.	xC_3	yV_3	zM_3	$xC_3+yV_3+zM_3$	zM
Σ	$xC=x\sum_{n=1}^{3} C_n$	$yV=y\sum_{n=1}^{3} V_n$	$zM=z\sum_{n=1}^{3} M_n$	$xC+yV+zM$	$xC+yV+zM$

Multipliers $x;y;z$ connect "values" and "prices". "Value" of output is so-called "labor cost" of output (the labor expended during the production process and embodied in commodities). "Value" of department' output is not equal in general case to the sum of "values" of commodities consumed in department during the production process (so-called "labor commanded").

$$\underbrace{C_1+C_2+C_3=C}_{\text{"labor cost"}} \neq \underbrace{C_1+V_1+M_1}_{\text{"labor commanded"}} \qquad (1)$$

$$\underbrace{V_1+V_2+V_3=V}_{\text{"labor cost"}} \neq \underbrace{C_2+V_2+M_2}_{\text{"labor commanded"}} \qquad (2)$$

$$\underbrace{M_1+M_2+M_3=M}_{\text{"labor cost"}} \neq \underbrace{C_3+V_3+M_3}_{\text{"labor commanded"}} \qquad (3)$$

These relations in price-terms can be rewritten as follows:

$$\underbrace{x\left(C_1+C_2+C_3\right)=xC}_{\text{"price of output"}} \neq \underbrace{xC_1+yV_1+zM_1}_{\substack{\text{"price of goods consumed}\\\text{during production process"}}} \qquad (4)$$

$$\underbrace{y\left(V_1+V_2+V_3\right)=yV}_{\text{"price of output"}} \neq \underbrace{xC_2+yV_2+zM_2}_{\substack{\text{"price of goods consumed}\\\text{during production process"}}} \qquad (5)$$

$$\underbrace{z\left(M_1+M_2+M_3\right)=zM}_{\text{"price of output"}} \neq \underbrace{xC_3+yV_3+zM_3}_{\text{"price of goods consumed during production process"}} \qquad (6)$$

Adam Smith (1776) separated two senses of term "labor": "labor cost" and "labor commanded".

Passage about "labor commanded":

"Every man is rich or poor according to the degree in which he can afford to enjoy the necessaries, conveniencies, and amusements of human life. But after the division of labour has once thoroughly taken place, it is but a very small part of these with which a man's own labour can supply him. The far greater part of them he must derive from the labour of other people, and he must be rich or poor according to the quantity of that labour which he can command, or which he can afford to purchase. The value of any commodity, therefore, to the person who possesses it, and who means not to use or consume it himself, but to exchange it for other commodities, is equal to the quantity of labour which it enables him to purchase or command. Labour, therefore, is the real measure of the exchangeable value of all commodities…
Wealth, as Mr. Hobbes says, is power… The power which that possession immediately and directly conveys to him [person], is the power of purchasing; a certain **command over all the labour***, or over all the produce of labour which is then in the market. His fortune is greater or less, precisely in proportion to the extent of this power; or to the quantity either of other men's labour, or, what is the same thing, of the produce of other men's labour, which it enables him to purchase or command. The exchangeable value of every thing must always be precisely equal to the extent of this power which it conveys to its owner" (Smith (1776), v.I, ch.5).*

Passage about "labor cost":

"The real price of every thing, what every thing really **costs** *to the man who wants to acquire it, is the toil and trouble of acquiring it… What is bought with money or with goods is purchased by labour, as much as what we acquire by the toil of our own body. That money or those goods indeed save us this toil. They contain the value of a certain quantity of labour which we exchange for what is supposed at the time to contain the value of an equal quantity. Labour was the first price, the original purchase-money that was paid for all things.*
…Though labour be the real measure of the exchangeable value of all commodities, it is not that by which their value is commonly estimated. It is often difficult to ascertain the proportion between two different quantities of labour. The time spent in two different sorts of work will not always alone determine this proportion. The different degrees of hardship endured, and of ingenuity exercised, must likewise be taken into account. There may be more labour in an hour's hard work than in two hours easy business; or in an hour's application to a trade which it cost ten years labour to learn, than in a month's industry at an ordinary and obvious employment. But it is not easy to find any accurate measure either of hardship or ingenuity. In exchanging indeed the different productions of different sorts of labour for one another, some allowance is commonly made for both…" (Smith (1776), v.I, ch.5)

We will demonstrate that this distinction between "labor cost" and "labor commanded" is very important point for true understanding of transformation problem. "Labor cost" coincides with "labor commanded" only if producers are exchanging goods on the base of "values".

We will consider in sections I - IX economy in which "non-trivial balance conditions" (NTBC) are fulfilled. Marx in chapter XX(4) of the second volume of "Capital" has introduced NTBC for the Model-2 but Model-2 can be converted into the Model-1 by means of new definition of the second and third departments. We will discuss in details economical sense and Marx's non-trivial conditions in section VI and X. We argue that NTBC were carried out in early capitalist economy.

Let's formulate NTBC mathematically:

A. NON-TRIVIAL BALANCE-CONDITIONS IN THE MODEL-1:

(A1) $xC_2 = yV_1$

(A2) $xC_3 = zM_1$

(A3) $yV_3 = zM_2$

B. NON-TRIVIAL BALANCE-CONDITIONS IN THE MODEL-2:

(B1) $xC_2 = y(V_1 + M_{1V})$

(B2) $xC_3 = yM_{1m}$

(B3) $y(V_3 + M_{3V}) = zM_{2m}$

Here the following designations are used:

$V_n; C_n$ - variable and constant capital in department n;

M_{nV} - expenditures of capitalists on "necessities of life" in department n;

M_{nm} - expenditures of capitalists on "luxury goods" in department n;

M_n - the total expenditures of capitalists in department n.

Relations (A) determine equilibrium price-vector which depends on arbitrary multiplicative positive constant.

Non-zero solution of system (A) exists if only determinant of matrix of coefficients equals zero.

$$\begin{vmatrix} C_2 & -V_1 & 0 \\ C_3 & 0 & -M_1 \\ 0 & V_3 & -M_2 \end{vmatrix} = 0 \tag{7}$$

It gives us the following equivalent equalities:

$$\frac{M_1}{M_2} = \frac{C_3 V_1}{C_2 V_3}; \tag{8}$$

$$\frac{m_1}{m_2} = \frac{V_2 C_3}{V_3 C_2} = \frac{k_2}{k_3}. \tag{9}$$

We will use the following designations:

$m = \frac{M}{V}$ - rate of surplus-value;

$k = \frac{V}{C}$ - "organic composition of the capital"[4].

Multipliers $x; y; z$ satisfy to the following relations:

$$t = \frac{x}{y} = \frac{V_1}{C_2} \tag{10}$$

$$y = \frac{M_2 z}{V_3} \tag{11}$$

[4] Although such definition of "organic composition" (V : C) differs from traditional definition (C : V) we will use the name "organic composition" for the ratio: k = V : C.

$$x = ty = \frac{M_2 V_1 z}{V_3 C_2} \qquad (12)$$

Solution of system (A) depends on arbitrary positive value z. This is the only arbitrary variable. Other two variables y and z are defined by means of relations (11) – (12). Consequently equilibrium prices of balanced exchange follow from non-trivial balance conditions (A).

We consider two partial cases: exchange on the base of "values" and exchange on the base of "prices of production". Each case corresponds to the some "value-structure" of economy.

CASE №1. Exchange on the base of "values" – value-structure (SV);

CASE №2. Exchange on the base of "prices of production" – value-structure (SP).

III. Value-composition of economy.

Value-structure of economy can be described by means of "labor cost" and "labor commanded" forms. "Value-composition" of economy comprises both these forms.

Table 3. Matrix-III of "value-composition" for the model-1 of simple production.

	Matrix-III(1). Goods consumed in each department (labor commanded aspect).				
	C (means of production) (1)	V (necessities of life) (2)	M (luxury goods) (3)	m	W (value of goods consumed in each department) (5)
I	C_1	$\beta(C-C_1)$	M_1	m_1	$C_1+\beta(C-C_1)(1+m_1)$
II	C_2	$\beta(V-C_2)$	M_2	m_2	$C_2+\beta(V-C_2)(1+m_2)$
III	C_3	$\beta(M-C_3)$	M_3	m_3	$C_3+\beta(M-C_3)(1+m_3)$
Σ	$C=C_1+C_2+C_3$	$V=\beta(V+M)$	$M=\frac{1-\beta}{\beta}V$		

	Matrix-III(2). Embodied labor in product (labor costs aspect)				
	C (transferring labor)	$\widehat{V}$ (necessary labor)	$\widehat{M}$ (surplus labor)	m	$\widehat{W}$ (labor cost of product)
I	C_1	$\alpha(C-C_1)$	$(1-\alpha)(C-C_1)$	m	C
II	C_2	$\alpha(V-C_2)$	$(1-\alpha)(V-C_2)$	m	V
III	C_3	$\alpha(M-C_3)$	$(1-\alpha)(M-C_3)$	m	M
Σ	C				

"Labor values" in labor theory of value coincide with "labor costs". "Necessary labor" is proportional to labor added during production process. Consequently the rates of surplus value in each department are equal (matrix III(2)). Employees of each industry obtain wage which is proportional to their labor. Matrix III(2) describes equilibrium state on labor-market. Labor added during production process (year) is equal $L=\widehat{V}+\widehat{M}$. Surplus-value $\widehat{M}$ in the Model-1 is equal to value of "luxury goods" consumed during year:

$$\widehat{M} = (1-\alpha)(V+M) = M \qquad (13)$$

We find (from matrix III(1) and formula (13)):

$$\alpha = \beta \quad (14)$$

Matrix III describes the simple reproduction since value of product of each department (last column in matrix III(2)) is equal to value of product consumed by all departments (last raw in matrix III(1)). Rate of surplus value in matrix III(2) and parameter $0 < \beta < 1$ are connected by the following relations.

$$m = \frac{1-\beta}{\beta} \qquad \beta = \frac{1}{1+m} \quad (15)$$

"Rates of surplus value" in matrix III(1) can differ from the true surplus-rate (15).

$$m_n = \frac{M_n}{V_n} \quad (16)$$

These "rates" satisfy to the following formula:

$$mV = \sum_{n=1}^{3} m_n V_n \quad (17)$$

Solution of system (A) must satisfy to relations (13)-(17).

IV. Exchange based on "values" (SV-structure)[5].

If goods are exchanging on the base of values then we have relation:

$$x = y = z = 1 \quad (18)$$

Balanced prices (A) coincide with prices of exchange on the base of values.

"Value-structure" ("labor commanded" aspect) has the following form in this case:

Table 4. Matrix-IV of value-structure for the exchange on the base of "values" (SV-structure).

	C	V	M	Σ
I.	$(1-b)aC$	$(1-b)(1-a)\cdot C$	bC	C
II.	$(1-b)(1-a)C$	YC	XC	$V = kC$
III.	bC	XC	ZC	M
Σ	C	$V = kC$	$M = mV$	$C+V+M$

The next formulas are fulfilled in this case:

$$\beta = \frac{V_1}{C - C_1} = \frac{(1-b)(1-a)}{1-a(1-b)} \quad (19)$$

$$m = \frac{M}{V} = \frac{b}{(1-a)(1-b)} \quad (20)$$

$$X = \frac{b\left[k-(1-b)(1-a)\right]}{1-a(1-b)} \quad (21)$$

$$Y = \frac{\left[k-(1-b)(1-a)\right](1-b)(1-a)}{1-a(1-b)} \quad (22)$$

[5] **Supplements** contain numerical examples of (SV) and (SP) structures and results of computations on the base of formulas of this paper. **Technical Excel-file** contains programs of calculations.

$$Z=\frac{b^2\left[k-(1-b)(1-a)\right]}{\left[1-a(1-b)\right](1-a)(1-b)} \tag{23}$$

$$m=m_1=m_2=m_3=\frac{M}{V}=\frac{X}{Y}=\frac{Z}{X}=\frac{b}{(1-a)(1-b)} \tag{24}$$

$$k=\frac{V}{C} \tag{25}$$

$$k_2=\frac{V_2}{C_2}=\frac{V_3}{C_3}=k_3=\frac{k-(1-b)(1-a)}{1-a(1-b)} \tag{26}$$

$$r_2=\frac{M_2}{C_2+V_2}=\frac{M_3}{C_3+V_3}=r_3=\frac{b\left[k-(1-a)(1-b)\right]}{(1-a)(1-b)(k+b)} \tag{27}$$

$$r=\frac{M}{C+V}=\frac{mk}{1+k}=\frac{bk}{(1-a)(1-b)(1+k)} \tag{28}$$

Let's substitute these relations into the matrix of value-structure.

Table 5. Matrix-IV(1) of value-structure ("labor commanded" aspect) for the exchange based of "values" (SV-structure).

	C	V	M	Σ
I.	$(1-b)aC$	$(1-b)(1-a)\cdot C$	bC	C
II.	$(1-b)(1-a)C$	$\frac{\left[k-(1-b)(1-a)\right](1-b)(1-a)}{1-a(1-b)}C$	$\frac{b\left[k-(1-b)(1-a)\right]}{1-a(1-b)}C$	$V=kC$
III.	bC	$\frac{b\left[k-(1-b)(1-a)\right]}{1-a(1-b)}C$	$\frac{b^2\left[k-(1-b)(1-a)\right]}{\left[1-a(1-b)\right](1-a)(1-b)}C$	M
Σ	C	$V=kC$	$M=mV$	

Matrix IV describes the possible value-structures which are consistent with the exchange based on "values" in the Model-1 of simple production. Any other value-structures are inconsistent with exchange based of values. Price-vector for value-structure IV is vector of values. As a rule vector of values differs from vector of prices of production.

Partial case is possible when balanced prices coincide both with values and prices of production. In this case we have the following value-structure.

Table 6. Matrix-V of value-structure ("labor commanded" aspect) for the exchange in which balanced prices = values = prices of production.

	C	V	M	Σ
I.	$(1-b)aC$	$(1-b)(1-a)\cdot C$	bC	C
II.	$(1-b)(1-a)C$	$\frac{(1-a)^2(1-b)}{a}C$	$\frac{b(1-a)}{a}C$	$V=kC$
III.	bC	$\frac{b(1-a)}{a}C$	$\frac{b^2}{a(1-b)}C$	mV
Σ	C	$V=kC=\frac{1-a}{a}C$	$M=mV=\frac{b}{a(1-b)}C$	$C+V(1+m)$

V. Exchange based on "prices of production" (SP-structure).

Let's find value-structures which are consistent with exchange based on the prices of production. "Non-trivial balance conditions" (A) in this case can be rewritten as follows:

(A'1) $C_2x=V_1y$

(A'2) $C_3x=r(C_1x+V_1y)=M_1z$

(A'3) $V_3y=r(C_2x+V_2y)=M_2z$

(A'4) $M_3z=r(C_3x+V_3y)$

The relation (B4) follows from (B1)-(B3) and definition of profit rate:

$$r=\frac{zM}{xC+yV} \tag{29}$$

We have symmetric matrix of input-flows.

Table 7. Matrix-VI of input-flows for the exchange based on the prices of production.

	C	V	M	Σ:
I.	C_1x	V_1y	$r(C_1x+V_1y)$	Cx
II.	C_2x	V_2y	$r(C_2x+V_2y)$	Vy
III.	C_3x	V_3y	$r(C_3x+V_3y)$	Mz
Σ:	Cx	Vy	$Mz=r(Cx+Vy)$	$C+V(1+m)$

The system (B) has nonzero solution if only the following relations are fulfilled:

$$k=\frac{C}{V}=\frac{C_1+C_2}{V_1+V_2}=\frac{C_3}{V_3}=k_3 \tag{30}$$

$$r=\frac{C_3}{C_1+C_2}=\frac{V_3}{V_1+V_2} \tag{31}$$

The solution of system (B) depends on one arbitrary parameter (z for example):

$$t\equiv\frac{x}{y}=\frac{V_1}{C_2} \tag{32}$$

$$x = \frac{V_1}{C_2} y \tag{33}$$

$$y = \frac{M_3}{r\left(C_3 t + V_3\right)} z \tag{34}$$

Value-structure ("labor commanded" aspect) has the following form in this case.

Table 8. Matrix-VII of value-structure for the exchange based on prices of production.

	C	V	M	Σ
I.	$(1-b)aC$	$kd(1-b)\cdot C$	$m_1 kd(1-b)C$	C
II.	$(1-b)(1-a)C$	$k(1-d)(1-b)C$	$m_2 k(1-d)(1-b)C$	$V = kC$
III.	bC	kbC	$m_3 kbC$	M
Σ	C	$V = kC$	$M = mV$	$C+V+M$

Parameters of this structure are connected by means of the following relations:

$$d = \frac{1-a(1-b)}{k+b} \tag{35}$$

$$\beta = \frac{k(1-b)}{k+b} \tag{36}$$

$$m = \frac{b(1+k)}{k(1-b)} \tag{37}$$

$$m_3 = \frac{m-(1-b)\left[m_1 d + m_2(1-d)\right]}{b} = m \tag{38}$$

$$\frac{m_2}{m_1} = \frac{(k+b)(1-a)}{k-(1-a)(1-b)} \tag{39}$$

$$t = \frac{kd}{1-a} \tag{40}$$

$$m_1 = \left(\frac{b}{1-b}\right)\cdot\frac{(k+b)(1+k)}{k\left[b+(1-a)(1+k)\right]} \tag{41}$$

$$r = \frac{b}{1-b} \tag{42}$$

Let's substitute these formulas into matrix VII.

Table 9. Matrix-VII(1) of value-structure for the exchange based on prices of production (SP-structure).

	C	V	M	Σ
I.	$(1-b)aC$	$\frac{k(1-b)[1-a(1-b)]}{k+b}\cdot C$	$\frac{b(1+k)[1-a(1-b)]}{[b+(1-a)(1+k)]}C$	C
II.	$(1-b)(1-a)C$	$\frac{k(1-b)[k-(1-a)(1-b)]}{k+b}C$	$\frac{b(1-a)(1+k)(k+b)}{b+(1-a)(1+k)}C$	$V=kC$
III.	bC	kbC	$\frac{b^2(1+k)}{1-b}C$	M
Σ	C	$V=kC$	$M=mV$	$C+V+M$

We see that SP-structure differs from SV-structure. These structures coincide if only value-structure has the form of matrix V.

Let's list Marx's Rules of Transformation:

Rule I. Aggregate surplus-value is equal to aggregate profit:

(RI) $M=r(Cx+Vy)$

Rule 2. Value of gross output is equal to price of production of gross output.

(RII) $C+V+M=(1+r)(Cx+Vy)$

Rule III. Rates of profit calculated on the base of values and prices of production are equal.

(RIII) $\frac{M}{C+V}=\frac{Mz}{Cx+Vy}$

Let $z=1$. The rule (RIII) follows from (RI) and (RII) in this case.

Let's prove that both first and second rules of transformation are fulfilled simultaneously for SP-structure. Substitute relations (35) – (41) into formulas (32) – (34).

$$y=\frac{(1+k)(1-a)(k+b)}{k[1-a+b+k(1-a)]} \tag{43}$$

$$t=\frac{k[1-a(1-b)]}{(k+b)(1-a)} \tag{44}$$

$$x=ty=\frac{(1+k)[1-a(1-b)]}{1-a+b+k(1-a)} \tag{45}$$

Rules (RI)-(RIII) follow from (43)-(45), (37) and (42):

$$Cx+Vy=C(x+ky)=C\left(\frac{(1+k)[1-a(1-b)+(1-a)(k+b)]}{1-a+b+k(1-a)}\right)=C(1+k)=C+V \tag{46}$$

$$r(Cx+Vy)=r(C+V)=\frac{b(1+k)C}{1-b} \tag{47}$$

$$M=mV=mkC=\frac{b(1+k)kC}{k(1-b)}=\frac{b(1+k)C}{1-b}=r(Cx+Vy) \tag{48}$$

We obtain the following structure for the economy of simple production in prices of production after substitution of formulas (43) and (45) into Table 7.

Table 10. Structure of economy with simple production and the exchange on the base of "production prices". Model-1.

	C (in "prices of production")	V (in "prices of production")	M (in "prices of production")
I.	$\frac{(1-b)a(1+k)[1-a(1-b)]}{1-a+b+k(1-a)}C$	$\frac{(1-b)[1-a(1-b)](1+k)(1-a)}{[1-a+b+k(1-a)]}C$	$\frac{b(1+k)[1-a(1-b)]}{[b+(1-a)(1+k)]}C$
II.	$\frac{(1-b)(1-a)(1+k)[1-a(1-b)]}{1-a+b+k(1-a)}C$	$\frac{(1-b)[k-(1-a)(1-b)](1+k)(1-a)}{[1-a+b+k(1-a)]}C$	$\frac{b(1-a)(1+k)(k+b)}{b+(1-a)(1+k)}C$
III.	$\frac{b(1+k)[1-a(1-b)]}{1-a+b+k(1-a)}C$	$\frac{b(1+k)(1-a)(k+b)}{[1-a+b+k(1-a)]}C$	$\frac{b^2(1+k)}{1-b}C$

VI. Non-trivial conditions of balanced exchange. Discussion.

So, Marx's transformation rules (RI)-(RIII) are fulfilled in the Model-1 if we take into account non-trivial balance conditions (NTBC). Matrix of input-flows has symmetric form in this case. Let's demonstrate that NTBC follow from Marx's analysis of simple reproduction in the second volume of "Capital", section XX(IV).

Let's consider this point in details. Marx considers model-2: I – "means of production", IIa – "necessities of life", and IIb – "luxury goods". Working-class bye only necessities of life whereas capitalists buy both necessities of life and luxury goods. Income of working-class is a wage. Income of capitalists is profit. So, Marx's model is our Model-2. Marx considers the following numerical example for the exchange between sub-divisions IIa (production of necessities of life) and IIb (production of luxury goods).

$$\begin{aligned} &\text{II}\, a:\ 400_V + 400_m \\ &\text{II}\, b:\ 100_V + 100_m \end{aligned} \qquad (49)$$

He writes:

"The labourers of IIb have received 100 in money as payment for their labour-power, or say £100. With this money they buy articles of consumption from capitalists IIa to the same amount. This class of capitalists buys with the same money £100 worth of the IIb commodities, and in this way the variable capital of capitalists IIb flows back to them in the form of money.

In IIa there are available once more 400_v in money, in the hands of the capitalists, obtained by exchange with their own labourers. Besides, a fourth of the part of the product representing surplus-value has been transferred to the labourers of IIb, and in exchange IIb (100_v) have been received in the form of articles of luxury".

Consequently capitalists IIa bought luxury goods in sum £100. They obtained these money owing to sale in sum £100 of own product (consumer necessities) to "labourers" IIb. We see that value of consumer necessities bought by IIb is equal to value of luxury goods bought by IIa.

"Now, assuming that the capitalists of IIa and IIb divide the expenditure of their revenue in the same proportion between necessities of life and luxuries — three-fifths for necessities for instance and two-fifths for luxuries — the capitalists of sub-class IIa will spend three-fifths of their revenue from surplus-value,

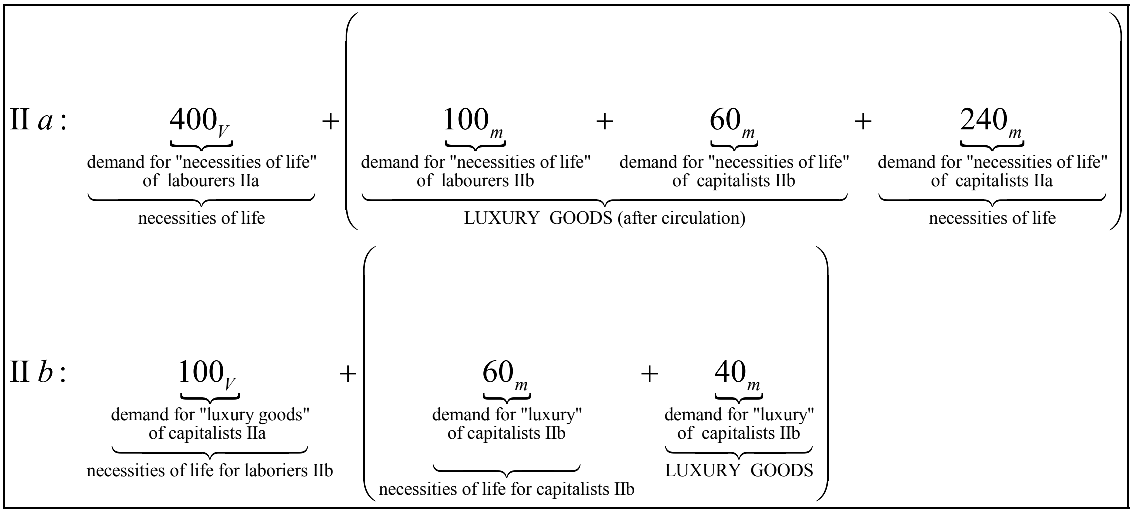
amounting to 400_s, or 240, for their own products, necessities of life, and two-fifths, or 160, for articles of luxury. The capitalists of sub-class IIb will divide their surplus-value of 100_s in the same way: three-fifths, or 60, for necessities, and two-fifths, or 40, for articles of luxury, the latter being produced and exchanged in their own sub-class".

Schema I (see below) illustrates results of exchange between IIa and IIb.

Marx writes:

"That in the case of annual product which is reproduced in the form of articles of consumption, the variable capital v advanced in the form of money can be realised by its recipients, inasmuch as they are labourers producing luxuries, only in that portion of the necessities of life which embodies for their capitalist producers prima facie their surplus-value; hence that ***v, laid out in the production of luxuries, is equal in value to a corresponding portion of s produced in the form of necessities of life****, and hence must be smaller than the whole of this s, namely $(IIa)_s$, and that the variable capital advanced by the capitalist producers of luxuries returns to them in the form of money only by means of the realisation of that v in this portion of s. This phenomenon is quite analogous to the realisation of $I_{(v+s)}$ in II_c, except that in the second case $(IIb)_v$ realizes itself in a* part *of $(IIa)_s$ of the same value. These proportions remain qualitatively determinant in every distribution of the total annual product, since it actually enters into the process of the annual reproduction brought about by circulation".*

Schema I. Input-flows of goods after circulation (exchange) between sub-divisions IIa and IIb.



Marx indicates that value of input-flow of "luxury" in sub-division IIa is equal to value of input-flow of "necessities of life" in sub-division IIb (160 = 160). Goods of sub-division IIa ("necessities of life") are exchanged onto the goods of sub-division IIb ("luxury") on the base of equilibrium price of balanced exchange (value = price of production in Marx's example). Marx's rule of balanced exchange between sub-divisions IIa and IIb means that price of goods exchanged one another ("luxury" for "necessities of life") is the same. It means that non-trivial balance-condition (B3): $y\left(V_3 + M_{3V}\right) = zM_{2m}$ is carried out in Marx's numerical example. This rule in the Model-1 (where capitalists buy only "luxury" and consequently $M_{2m} = M_2$ and $M_{3V} = 0$) can be formulated as balance-condition (A3)[6]:

(A3) $zM_2 = yV_3$

Marx's numerical example including department I is presented by Schema II.

[6] Balance-conditions (A1) and (A2) follow from (A1) in the model-1 and balance-conditions (B1) and (B2) follow from (B1) in the model-2 if the economy satisfies to conditions of simple reproduction.

Schema II. Marx's numerical example.

$$\text{I.}\ \ 4000_C + 1000_V + 1000_m \ \text{- means of production}$$

$$\text{IIa.}\ 1600_C + 400_V + 400_m \ \text{- necessities of life}$$

$$\text{IIb.}\ 400_C + 100_V + 100_m \ \text{- luxury}$$

Capitalists of sub-division IIa buy "luxury" in sum 160. Capitalists and laborers of subdivision IIb buy "necessities of life" in sum 160 also. Capitalists IIa consume own product ("necessities of life") in sum 240. Capitalists IIb consume own product ("luxury") in sum 40. We have the following Schema III.

Schema III. Realization of profit in sub-divisions IIa and IIb.

$$400_m(\text{IIa}) = \underbrace{160_m}_{\text{LUXURY}} + \underbrace{240_m}_{\text{NECESSITIES OF LIFE}}$$

$$100_m(\text{IIb}) = \underbrace{40_m}_{\text{LUXURY}} + \underbrace{60_m}_{\text{NECESSITIES OF LIFE}}$$

Let's substitute this composition into Schema II.

Schema IV. Matrix of input-flows.

$$\text{I.}\ \ 4000_C + 1000_V + 1000_m \ \text{- means of production}$$

$$\text{IIa.}\ 1600_C + \underbrace{400_V}_{\text{NECESSITIES OF LIFE}} + \left(\underbrace{240_m}_{\text{NECESSITIES OF LIFE}} + \underbrace{160_m}_{\text{LUXURY}} \right) \ \text{- necessities of life}$$

$$\text{IIb.}\ 400_C + \underbrace{100_V}_{\text{NECESSITIES OF LIFE}} + \left(\underbrace{60_m}_{\text{NECESSITIES OF LIFE}} + \underbrace{40_m}_{\text{LUXURY}} \right) \ \text{- luxury}$$

Value of "luxury" consumed in sub-division IIa is equal to value of "necessities of life" consumed in sub-division IIb. Rearrange Schema IV as follows:

Schema V. Matrix of input-flows.

$$\text{I.}\ \ 4000_C + 1000_V + 1000_m \ \text{- means of production}$$

$$\text{IIa.}\ 1600_C + \underbrace{640_{V+m}}_{\text{NECESSITIES OF LIFE}} + \underbrace{160_m}_{\text{LUXURY}} \ \text{- necessities of life}$$

$$\text{IIb.}\ 400_C + \underbrace{160_{V+m}}_{\text{NECESSITIES OF LIFE}} + \underbrace{40_m}_{\text{LUXURY}} \ \text{- luxury}$$

Exchange between sub-division IIa and department I is analogous. Capitalists IIa buy "means of production" in sum 1600. Capitalists and laborers I buy "necessities of life" in sum 1600 (1000 – laborers I and 600 – capitalists I). Capitalists IIb buy "means of production" in sum 400. Capitalists I buy "luxury" in the sum 400 and "necessities of life in sum 600.
Finally, we have the following complete Schema of input-flows for Marx's example.

Schema VI. Matrix of input-flows in Marx's numerical example.

$$\text{I.}\quad \underbrace{4000_C}_{\text{MEANS OF PRODUCTION}} + \underbrace{1600_{V+m}}_{\text{NECESSITIES OF LIFE}} + \underbrace{400_m}_{\text{LUXURY}} \quad \text{- means of production}$$

$$\text{IIa.}\quad \underbrace{1600_C}_{\text{MEANS OF PRODUCTION}} + \underbrace{640_{V+m}}_{\text{NECESSITIES OF LIFE}} + \underbrace{160_m}_{\text{LUXURY}} \quad \text{- necessities of life}$$

$$\text{IIb.}\quad \underbrace{400_C}_{\text{MEANS OF PRODUCTION}} + \underbrace{160_{V+m}}_{\text{NECESSITIES OF LIFE}} + \underbrace{40_m}_{\text{LUXURY}} \quad \text{- luxury}$$

We see that all non-trivial balance-conditions are fulfilled. Matrix of input-flows in prices of balanced exchange is symmetric.

Marx's model-2 can be transformed into model-1 owing to redefinition of sub-divisions IIa and IIb. NTBC will be fulfilled in the model-2 if non-trivial balance-conditions are carried out in the model-2. We discuss in details in Section XI the **NTBC-invariant conversion** of models of capitalist simple production: model-4 → model-2 → other model-2 → model-1 (as partial case of model-2: $M_{1V} = M_{2V} = M_{3V} = 0$).

Marx emphasizes that concrete figures and proportions in his numerical example can be chosen arbitrary. Consequently Marx' general logical conclusions (NTBC-rules) shouldn't depend on a choice of concrete figures in his numerical example:

*"**What is arbitrary here is the ratio of the variable to the constant capital of both I and II and so is the identity of this ratio for I and II and their sub-divisions. As for this identity, it has been assumed here merely for the sake of simplification, and it would not alter in any way the conditions of the problem and its solution if we were to assume different proportions.** However, the necessary result of all this, on the assumption of simple reproduction, is the following…*

In the exchange established above of (IIb)$_v$ for a portion of (IIa)$_s$ of the same value, and in the further exchanges between (IIa), and (IIb), it is by no means assumed that either the individual capitalists of IIa and IIb or their respective totalities divide their surplus-value in the same proportion between necessary articles of consumption and articles of luxury. The one may spend more on this consumption, the other more on that. On the basis of simple reproduction it is merely assumed that a sum of values equal to the entire surplus-value is realised in the consumption-fund. The limits are thus given. Within each department the one may spend more in a, the other in b". (vol. II; chapter XX(4))

Marx's numerical example corresponds to early capitalist economy in which credit plays very minor role. Marx writes. It follows from the next passages:

"It goes without saying that this applies only to the extent that it all is really a result of the process of reproduction itself, i.e., to the extent that the capitalists of IIb, for instance, do not obtain money-capital for v on credit from others" (vol. II; chapter XX(4))

"…credit-production plays only a very minor role, or none at all, during the first epoch of capitalist production" (vol. II; chapter IV)

Why Marx postulated non-trivial balance-conditions in the early capitalist economy with simple production? The answer in details will be given in Section X of this paper where we consider more realistic model-4 of capitalist economy. NTBC in the model-4 follow from some essential peculiarities of merchant capitalism (early form of capitalism).

Shortly, the basic arguments are following.

The models -1 and 2 don't take into account the "labor" of capitalists-entrepreneurs although this labor should be included into "values" of goods in the same way as the "labor" of workers. Model-4 is more realistic model of capitalist simple production since this model takes into account "labor of capitalists". Solution of transformation problem in model-4 exists for arbitrary value-structure of economy (without imposition of NTBC-rules).

Marx's non-trivial balance-conditions are fulfilled automatically in model-4 if the following equalities are fulfilled:

(1) shares $\delta_n \equiv 1-\alpha_n$ of "necessities of life" in "net surplus-value" created in departments are equal ($\alpha_1 = \alpha_2 = \alpha_3$). "Net surplus-value" equals surplus-value minus "wage" of "capitalist-entrepreneur".

(2) shares γ_n of "wage of capitalist-entrepreneur" in full payment of labor (full wage of workers and capitalists) are equal in all departments ($\gamma_1 = \gamma_2 = \gamma_3$).

It is very probable that these conditions (1) and (2) were carried out in the early capitalist economy of merchant capitalism.

Handicraftsmen and peasants in medieval Europe were working at own homes and workshops by means of own equipment and instruments. Merchants were giving them row materials for processing (so-called dispersed manufactory) and monetary advance payments in exchange for their production. The capital of a merchant hasn't been fixed in any one field of activity. The capital of a merchant hasn't been embodied in buildings or in the equipment (as the capital of industrial capitalist) since handicraftsmen and peasants in the middle ages used as a rule own equipment and they performed work at home or inside of own workshops. The capital of merchants-businessmen could be easily enough transferred from one sphere of business to another sphere. Therefore merchants could invest money-capital in different projects: both into production of means of production and into production of subsistence goods and luxury. Often the capital of separate merchant was dispersed between three departments of economy. Each merchant union (or separate merchant) participated often in production of different products: "means of production"; "subsistence goods" and "luxury goods". Therefore the capital of each department consisted of separate capitals of many merchants-entrepreneurs. Although parameters α and γ for each merchant-capitalist differed one another the average values of parameters in the departments as a whole were almost equal ("the law of big numbers"). "The law of big numbers" was equalizing parameters α and γ of different departments. We prove in section IX that conditions $\alpha_1 = \alpha_2 = \alpha_3$ and $\gamma_1 = \gamma_2 = \gamma_3$ guarantee the execution of non-trivial balance-conditions in the model-4. We can conclude that non-trivial balance-conditions were executed at stage of early capitalism owing to high liquidity of merchant capital[7].

Supplement III contains results of numerical modelling. 1000 capitals K_n of arbitrary value ($K_n \le 1000$) were distributed randomly between three departments so that trivial conditions of balance are fulfilled with high precision. Although values α_n and γ_n of separate capitals don't coincide parameters α and γ of departments are coincide with good precision[8]. Lower Table of Supplement III describes result of transformation of "values" into "prices of production". Deviation from NTBC ~ 1%.

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[7] Many authors indicate that merchant-capitalists were indeed the main group of capitalist entrepreneurs in the Middle Ages. See monograph Fernand Braudel "Material Civilization" and works Banaji, J. (2003); Heaton, H. (1920); John, A. (1962); Jonathan, M.J. (2006); Mielants, E.H. (2007).

[8] See also technical Excel-file with program of computations: worksheets 'RandCap1'; RandCap2' and 'RandCap3'.

VII. Bortkiewicz' solution.

Bortkiewicz (1907a) considered Model-1. Let's remember his problem statement. He postulated the following relations[9]:

1) Trivial conditions of simple production:

$$C_1 + C_2 + C_3 = C_1 + V_1 + M_1$$
$$V_1 + V_2 + V_3 = C_2 + V_2 + M_2 \qquad (50)$$
$$M_1 + M_2 + M_3 = C_3 + V_3 + M_3$$

2) Conditions for value-structure of economy:

$$C_1 + V_1 + M_1 = C$$
$$C_2 + V_2 + M_2 = V \qquad (51)$$
$$C_3 + V_3 + M_3 = M$$

3) Conditions for surplus-value:

$$M_1 = mV_1$$
$$M_2 = mV_2 \qquad (52)$$
$$M_3 = mV_3$$

Relations (52) are fulfilled in "labor-costs"-structure only. Consequently M_n is surplus-value. Relations (50) – (52) describes "labor-costs"-structure in the economy with simple production. Symbols $M_1; M_2; M_3$ does not designate value of something goods. These are symbols which fix surplus-value in each department. We demonstrated earlier that value of goods consumed by capitalists of each department can differ from surplus-value of each department. These quantities coincide if only exchange of goods based on values. Consequently symbols $M_1; M_2; M_3$ don't mean any goods – "luxury" for example. These symbols designate surplus-value only. Bortkiewicz does not distinct "labor cost" and "labor commanded" aspects. Let's correct Bortkiewicz' relations (50) – (52) as follows:

1) Conditions of simple production:

$$C_1 + C_2 + C_3 = C$$
$$V_1 + V_2 + V_3 = V \qquad (50c)$$
$$\widehat{M}_1 + \widehat{M}_2 + \widehat{M}_3 = \widehat{M}$$

2) Conditions for value-structure of economy:

$$C_1 + V_1 + \widehat{M}_1 = C$$
$$C_2 + V_2 + \widehat{M}_2 = V \qquad (51c)$$
$$C_3 + V_3 + \widehat{M}_3 = \widehat{M}$$

3) Conditions for surplus-values:

$$\widehat{M}_1 = mV_1$$
$$\widehat{M}_2 = mV_2 \qquad (52c)$$
$$\widehat{M}_3 = mV_3$$

[9] Bortkiewicz used designation S for surplus-value.

Symbols $\widehat{M}_n$ designate surplus-value. It is necessary to take into account relations for "labor commanded" - structure in order to obtain complete problem statement. These are the following relations:

$$\underbrace{\left(\underbrace{M}_{\substack{\text{value}\\ \text{of "luxury"}}} = \underbrace{M_1}_{\substack{\text{"luxury"}\\ \text{consumed in}\\ \text{department I}}} + \underbrace{M_2}_{\substack{\text{"luxury"}\\ \text{consumed in}\\ \text{department II}}} + \underbrace{M_3}_{\substack{\text{"luxury"}\\ \text{consumed in}\\ \text{department III}}} \right)}_{\text{"LABOR COMMANDED"}} = \underbrace{\left(\underbrace{\widehat{M}}_{\text{surplus-value}} = \underbrace{\widehat{M}_1}_{\substack{\text{surplus-value}\\ \text{produced in}\\ \text{department I}}} + \underbrace{\widehat{M}_1}_{\substack{\text{surplus-value}\\ \text{produced in}\\ \text{department II}}} + \underbrace{\widehat{M}_1}_{\substack{\text{surplus-value}\\ \text{produced in}\\ \text{department III}}} \right)}_{\text{"LABOR COSTS"}} \quad (53)$$

$$\widehat{M} = mV = M = \sum_{n=1}^{3} M_n = \sum_{n=1}^{3} m_n V_n \quad (54)$$

$$\frac{m_1}{m_2} = \frac{V_2 C_3}{V_3 C_2} = \frac{k_2}{k_3} \quad (55)$$

Bortkiewicz has not taken into account the "labor-commanded-aspect" of value-composition and non-trivial balance-conditions in early capitalist economy. Consequently Bortkiewicz' problem statement was not complete.

Let's consider Bortkiewicz' equations for "production prices":

$$\begin{aligned} &(1+r)(C_1 x + V_1 y) = xC \\ &(1+r)(C_2 x + V_2 y) = yV \\ &(1+r)(C_3 x + V_3 y) = zM = zM_1 + zM_2 + zM_3 \end{aligned} \quad (56)$$

Let's impose yet non-trivial balance-conditions (for the exchange based on "production prices"):

(B1) $C_2 x = V_1 y$

(B2) $C_3 x = r(C_1 x + V_1 y) = M_1 z$

(B3) $V_3 y = r(C_2 x + V_2 y) = M_2 z$

(B4) $M_3 z = r(C_3 x + V_3 y)$

These are Marx's non-trivial balance-conditions for the model-1 with the exchange based on "prices of production". Relations (B) should be carried out in early capitalist economy with exchange based on "production prices". We illustrated in previous Section that relations (B) follow from Marx's schemas of simple reproduction. These relations impose condition of symmetry onto "input-flows matrix" of Model-1.

Bortkiewicz didn't take into account these non-trivial balance-conditions. Mathematically it means that he spread the set of possible solutions including even those solutions which don't satisfy to the conditions (B). Consequently Bortkiewicz considered wider set of "solutions". We proved (formulas (46)-(48)) that solution of transformation problem (for the economy of simple production with exchange based on "production prices" and non-trivial balance-conditions) exists always in the Model-1 (and in the Model-2 also – see below).

So, Bortkiewicz' solution doesn't take into account two aspects of task. First, Bortkiewicz omitted analysis of "labor commanded" aspect i.e. "labor-structure" which describes the proportions by which each department consumes the different articles of product produced by all departments. Second, Bortkiewicz omitted non-trivial balance-conditions (B) which follow from Marx's analysis of schemas of simple reproduction in the second volume of "Capital".

VIII. The problem of historical transformation.

We proved that Marx's rules of transformation are fulfilled in the Model-1 if exchange of goods based on "prices of production". Marx assumed that goods exchanged as "values" in the past (in pre-capitalist era). Each "value-structure" is connected with the definite equilibrium price-vector of balanced exchange. This price-vector coincides with values if "value-structure" of economy has the form of matrix IV (SV-structure) and it coincides with "production prices" if "value-structure" has the form of matrix VII (SP-structure). Consequently the process of **historical** transformation could take place if only value-structure (SV) transformed into value-structure (SP) during some "transition period". How such process could proceed?

Let's introduce "technological coefficients" γ_n as follows:

$$\gamma_1 = \frac{C_1}{C}; \qquad C_1 = C\gamma_1 \tag{57}$$

$$\gamma_2 = \frac{C_2}{V} \qquad C_2 = V\gamma_2 \tag{58}$$

$$\gamma_3 = \frac{C_3}{M} \qquad C_3 = M\gamma_3 \tag{59}$$

These quantities depend only on technologies of production in each department. The change of these coefficients indicates on change in technique and organization of labor. Value-composition (Table III) and some calculations give us the following relations:

$$\beta = \frac{1}{1+m} \tag{60}$$

$$V_1 = \frac{1-\gamma_1}{1+m} C \tag{61}$$

$$V_2 = \frac{1-\gamma_2}{1+m} V \tag{62}$$

$$V_3 = \frac{1-\gamma_3}{1+m} M \tag{63}$$

$$k_n = \frac{1-\gamma_n}{\gamma_n(1+m)} \qquad n = 1;2;3 \tag{64}$$

$$k = \frac{1-\gamma_1}{\gamma_2 + m\gamma_3} \tag{65}$$

$$r_n = \frac{m_n V_n}{C_n + V_n} = \frac{m_n(1-\gamma_n)}{1+\gamma_n m} \tag{66}$$

$$q_1 \equiv \frac{V_1}{V} = \frac{1-\gamma_1}{k(1+m)} \tag{67}$$

$$q_2 \equiv \frac{V_2}{V} = \frac{1-\gamma_2}{1+m} \tag{68}$$

$$q_3 \equiv \frac{V_3}{V} = \frac{m(1-\gamma_3)}{1+m} \tag{69}$$

Matrix of value-structure can be rewritten in new variables as follows:

Table 11. Matrix VIII of value-structure via technological coefficients.

	C	V	M	Σ
I.	$\gamma_1 C$	$\beta(1-\gamma_1)\cdot C$	$m_1\beta(1-\gamma_1)C$	C
II.	$\gamma_2 kC$	$\beta(1-\gamma_2)kC$	$m_2\beta(1-\gamma_2)kC$	$V = kC$
III.	$\gamma_3 mkC$	$\beta(1-\gamma_3)mkC$	$m_3\beta(1-\gamma_3)mkC$	$M = mkC$
Σ	C	$V = kC$	$M = mV = mkC$	

Let's consider three cases.

CASE №1. Exchange based on values ($x = y = z$) gives the following equalities:

$$\gamma_2 = \gamma_3 \quad (70)$$

$$m = m_1 = m_2 = m_3 \quad (71)$$

$$k_2 = k_3 \quad (72)$$

CASE №2. "Transition period". The next relations follow from conditions (A):

$$\frac{m_1}{m_2} = \frac{k_2}{k_3} = \frac{\gamma_3(1-\gamma_2)}{\gamma_2(1-\gamma_3)} \quad (73)$$

$$m = \frac{1-\beta}{\beta} \quad (74)$$

$$m = q_1 m_1 + q_2 m_2 + q_3 m_3 \quad (75)$$

Conditions (73) – (75) determine the set of possible value-structures during "transition period". Other quantities are equal:

$$k = \frac{1-\gamma_1}{\gamma_2 + m\gamma_3} \quad (76)$$

$$m_2 = \frac{m\left[1+m-m_3(1-\gamma_3)\right]k\gamma_2(1-\gamma_3)}{(1-\gamma_2)\left[\gamma_3(1-\gamma_1)+k\gamma_2(1-\gamma_3)\right]} = \frac{m\left[1+m-m_3(1-\gamma_3)\right]\gamma_2(1-\gamma_3)}{(1-\gamma_2)\left[\gamma_2+m\gamma_3^2\right]} \quad (77)$$

$$m_1 = \frac{\gamma_3(1-\gamma_2)}{\gamma_2(1-\gamma_3)}m_2 \quad (78)$$

Formulas (70) – (72) is partial case of general relations (73) – (78).

CASE №3. Exchange based on "production prices".
We have in this case the following relations:

$$\frac{C}{C_1 + C_2} = \frac{V}{V_1 + V_2} = \frac{M}{M_1 + M_2} = 1 + r \quad (79)$$

These relations give the following equalities:

$$k_3 = k \quad (80)$$

$$m_3 = m \quad (81)$$

Relations (80) – (81) are fulfilled if rate of surplus-value is equal:

$$m = \frac{\gamma_2\left(1-\gamma_3\right)-\gamma_3\left(1-\gamma_1\right)}{\gamma_3\left(\gamma_3-\gamma_1\right)} = \frac{\frac{\gamma_2}{\gamma_3}-\chi}{\chi-1} \quad (82)$$

$$\chi = \frac{1-\gamma_1}{1-\gamma_3} \quad (83)$$

Consequently all parameters in matrix VIII are functions of three technological coefficients $\gamma_1;\gamma_2;\gamma_3$. Parameters of matrices VII and VIII are interconnected as follows:

$$\gamma_1 = a\left(1-b\right) \quad (84)$$

$$\gamma_2 = \frac{\left(1-a\right)\left(1-b\right)}{k} \quad (85)$$

$$\gamma_3 = \frac{1-b}{1+k} \quad (86)$$

$$a = \frac{\gamma_1\left(\gamma_2-\gamma_3\right)}{\gamma_1\left(\gamma_2-\gamma_3\right)+\gamma_2\left(\gamma_3-\gamma_1\right)} \quad (87)$$

$$b = \frac{\gamma_2\left(1-\gamma_3\right)-\gamma_3\left(1-\gamma_1\right)}{\gamma_2-\gamma_3} \quad (88)$$

$$k = \frac{\gamma_3-\gamma_1}{\gamma_2-\gamma_3} \quad (89)$$

We see that initial exchange based on values supposes other value-structure than exchange based on production prices. (SV) and (SP) structures coincide when these value-structures has the same form of matrix V (Table V).

Process of historical transformation can be described theoretically as some “trajectory” of the economic system in 4-dimensional space $\left(m;\gamma_1;\gamma_2;\gamma_3\right)$. Two variants of historical transformation are possible theoretically in general case as it follows from formulas (82) – (83). These variants describe economy with positive rate of surplus-value. Technological coefficients after transformation must satisfy to one of two systems of inequalities:

VARIANT №1.

$$\gamma_1 < \gamma_2 \quad (90)$$

$$\gamma_1 < \gamma_3 < \frac{\gamma_2}{1+\left(\gamma_2-\gamma_1\right)} \quad (91)$$

VARIANT №2.

$$\gamma_1 > \gamma_2 \quad (92)$$

$$\gamma_1 > \gamma_3 > \frac{\gamma_2}{1+\left(\gamma_2-\gamma_1\right)} \quad (93)$$

We have conditions $\gamma_2 = \gamma_3$ and $m_3 = m$ for initial state of economy (Case №1). Transformation is possible if coefficients γ_n after transformation satisfy to inequalities (90) – (91) or to inequalities (92) – (93). Condition $m_3 = m$ is fulfilled both before transformation and after transformation. Let’s assume that this condition is fulfilled also during transition period: $m_3 = m$

There is some kind of a “barrier” which separates region in the space $(m;\gamma_1;\gamma_2;\gamma_3)$ where exchange based on production prices is possible. Relative change of parameter γ_3 must be larger than this “barrier”:

$$\left|\frac{\Delta\gamma_3}{\gamma_3}\right| \ge \left|\frac{\gamma_2-\gamma_1}{1+(\gamma_2-\gamma_1)}\right| \qquad (94)$$

Figure 1 illustrates position of this barrier in the plane $(\gamma_2;\gamma_3)$.

Balanced exchange in transition period differs both from exchange based on values and exchange based on prices of production. The following formulas for profit rates can be deduced from conditions (A):

$$1+r_1=\frac{Cx}{C_1x+V_1y}=\frac{Cx}{C_1x+C_2x}=\frac{C}{C_1+C_2} \qquad (95)$$

$$1+r_2=\frac{Vy}{C_2x+V_2y}=\frac{Vy}{V_1y+V_2y}=\frac{V}{V_1+V_2} \qquad (96)$$

$$1+r_3=\frac{Mz}{C_3x+V_3y}=\frac{Mz}{M_1z+M_2z}=\frac{M}{M_1+M_2} \qquad (97)$$

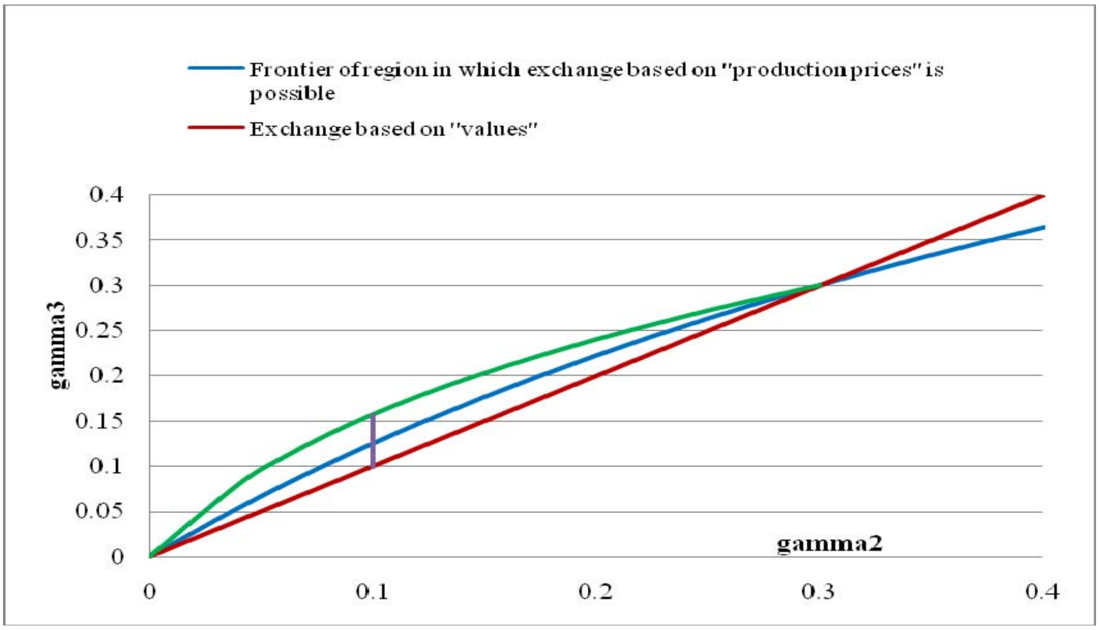
It gives us (accounting formulas (76) – (78) and (93)) the following relations:

$$r_1=\frac{m\gamma_3(1-\gamma_1)}{\gamma_2+m\gamma_1\gamma_3} \qquad (98)$$

$$r_2=\frac{m(1-\gamma_3)}{1+m\gamma_3} \qquad (99)$$

$$r_3=\frac{m(1-\gamma_3)}{1+m\gamma_3} \qquad (100)$$

Figure 1. Region of exchange based on production prices in the plane of technological coefficients $(\gamma_2;\gamma_3)$.



Historical transformation couldn't contradict to the aspiration of capitalists to increase the profit rate. We see from formulas (98) – (100) that rates $r_3; r_2$ decreases when technological coefficient γ_3 grows. Coefficient γ_3 is the share of means of production in value of product. Technological progress leads to the increase of this share (= economy of living labor). Therefore coefficients γ_n (and consequently rate of surplus-value as it follows from formulas (99) – (100)) should increase in the process of historical transformation.

We modeled this process for two variants of transformation (Variant №1 and Variant №2). Our model of historical transformation assumes that coefficients $\gamma_1; \gamma_2$ don't change whereas coefficients $\gamma_1; m$ can change. This is the model of historical transformation on account of technical progress in third department ("luxury goods"). Let γ_{3P} be new parameter γ_3 and m_P be new rate of surplus-value after transformation. We imposed condition $r_3 = Const$ during transition period (technical progress shouldn't decrease the rate of profit in modernized department). Formulas (98) – (100) and condition $r = r_1 = r_2 = r_3$ after transformation lead to the following relations:

$$\gamma_{3P} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \tag{101}$$

$$A = m(1-\gamma_3) - (1+m)(1+\gamma_2-\gamma_1) \tag{102}$$

$$B = \gamma_2(1+m) + (1+m\gamma_3)(1+\gamma_2-\gamma_1) - \gamma_1 m(1-\gamma_3) \tag{103}$$

$$C = -\gamma_2(1+m\gamma_3) \tag{104}$$

$$m_P = \frac{m(1-\gamma_3)}{(1-\gamma_{3P}) + m(\gamma_3-\gamma_{3P})} \tag{105}$$

$$r_{1P} = \frac{m_P\gamma_{3P}(1-\gamma_1)}{\gamma_2 + m_P\gamma_1\gamma_{3P}} = r_{2P} = r_{3P} = \frac{m_P(1-\gamma_{3P})}{1+m_P\gamma_{3P}} = r_3 = \frac{m(1-\gamma_3)}{1+m\gamma_3} \tag{106}$$

Figures 2 - 4 illustrate the growth of rate of surplus-value after transformation. We see that the increase m_P is moderate if $\gamma_2 = \gamma_3 < 0.4$ before transformation. We suppose that such figures are in the consistence with our knowledge about economy of pre-capitalist era in which level of technical development was very low. Significant changes in rate of surplus-value during transition period were practically impossible because both capitalists and workers struggle against any change of this rate. The Supplement 2 contains description of two models (VARIANT №1 and VARIANT №2).

Figure 2. The increase of rate of surplus-value after transition period (rate of profit in third department is fixed).

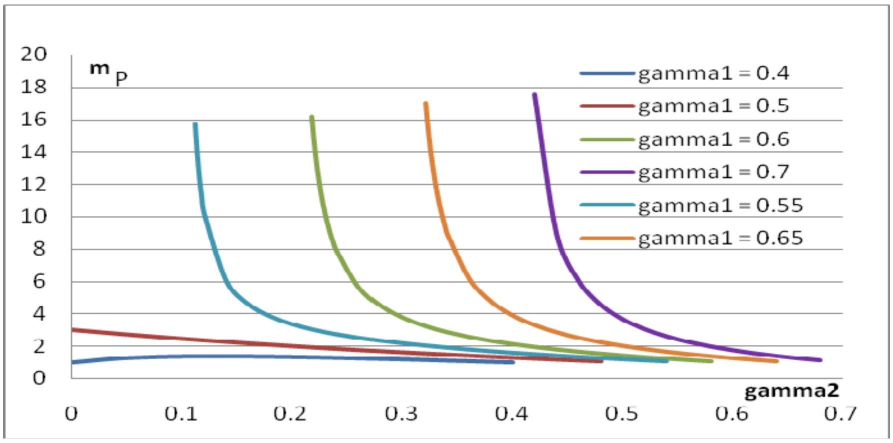


Figure 3. The increase of rate of surplus-value after transition period (rate of profit in third department is fixed).

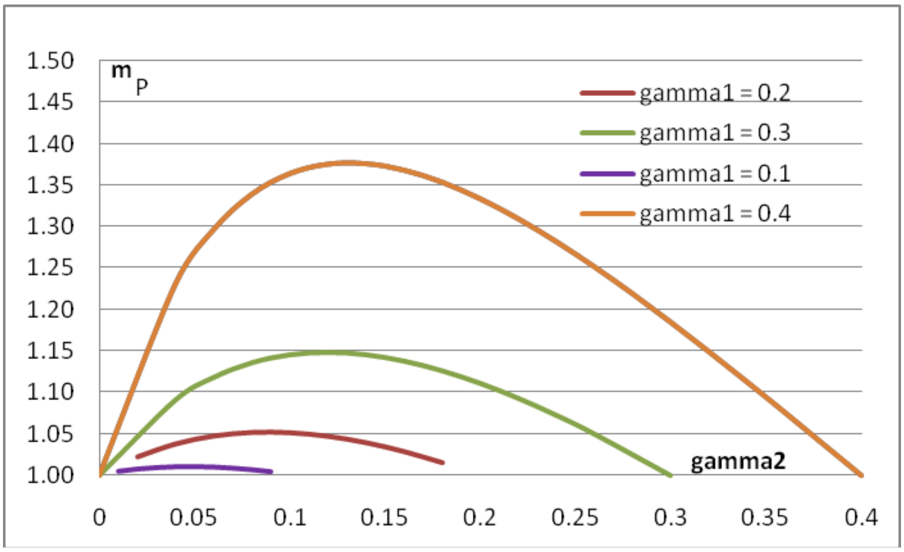
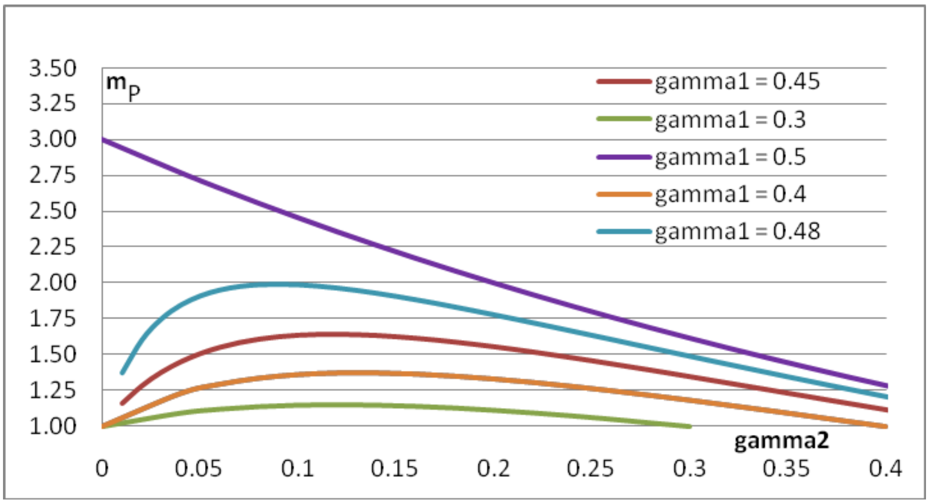


Figure 4. The increase of rate of surplus-value after transition period (rate of profit in third department is fixed).



IX. Current transformation of "values" into "prices" in the model-2.

Consider the general case when matrix of input flows is asymmetric. This is case when profit of capitalists is spent on both "luxury" and "consumer goods". Structure of economy in this case has the following form:

Table 12. Matrix of input-flows.

	C	V	M	W
I	xC_1	yV_1	$yM_{1V}+zM_{1m}$	xC
II	xC_2	yV_2	$yM_{2V}+zM_{2m}$	$y(V+M_V)$
III	xC_3	yV_3	$yM_{3V}+zM_{3m}$	zM_m
Σ	xC	yV	yM_V+zM_m	

Symbols M_V (M_m) designate value of "consumer goods" ("luxury") consumed by capitalists. Multipliers $(x;y;z)$ transfer "values" into "prices of production". The aggregates ($yM_{1V}+zM_{1m}$ and xC_3); ($yM_{2V}+zM_{2m}$ and yV_3) are not equal in this case (asymmetric matrix of input-flows).

Non-trivial conditions of balanced exchange for the simple production in this case can be formulated as follows:
(SI) Equations of non-trivial balance in "prices":

$$xC_2 = y(V_1 + M_{1V}) \tag{107}$$

$$xC_3 = zM_{1m} \tag{108}$$

$$y(V_3 + M_{3V}) = zM_{2m} \tag{109}$$

Second, input and output flows (in "values") in economy with simple reproduction satisfy to trivial balance-conditions:
(SII) Equations for input-output balance (in "values").

$$C_1 + C_2 + C_3 = C = C_1 + V_1(1+m) \tag{110}$$

$$V + M_V = C_2 + V_2(1+m) \tag{111}$$

$$M_m = C_3 + V_3(1+m) \tag{112}$$

Equation (112) follows from equations (110) – (111).
Third, we have condition for "prices of production": prices of balanced exchange coincide with "production prices":
(SIII) Condition for "prices of production":

$$yM_{nV} = \alpha_n r(C_n x + V_n y) \qquad n = 1;2;3 \tag{113}$$

$$zM_{nm} = (1-\alpha_n) r(C_n x + V_n y) \tag{114}$$

Finally, we postulate Marx's transformation rules:
(SIV) Transformation rules:

$$M = mV = M_V + M_m = yM_V + zM_m \tag{115}$$

$$C + V = Cx + Vy \tag{116}$$

Parameters of this model are connected by relations:

$$M_V = M_{1V} + M_{2V} + M_{3V} \tag{117}$$

$$M_m = M_{1m} + M_{2m} + M_{3m} \tag{118}$$

So, the statement problem in this case includes four systems of equations (SI) – (SIV). The solution depends on six arbitrary positive values: $C_1; C_2; C_3; V_1; V_2; \alpha_2$. Unknown variables of our system of equations (SI) – (SIV) are some functions of these parameters.
Solution is “realistic” if all parameters and variables are positive and economically reasonable.

SOLUTION OF THE SYSTEM (SI) – (SIV).

Equations (108), (109), (113), (114) give the following relations:

$$t = \frac{x}{y} = \frac{V_1}{C_2 - \beta_1 C_3}; \quad \beta_1 = \frac{\alpha_1}{1-\alpha_1}; \quad x = ty. \tag{119}$$

$$r = \frac{C_3 t}{(1-\alpha_1)(C_1 t + V_1)} = \frac{C_3}{(1-\alpha_1)(C_1 + C_2 - \beta_1 C_3)} = \frac{r_0}{(1-\alpha_1)(1-\beta_1 r_0)}; \qquad r_0 = \frac{C_3}{C_1 + C_2}. \tag{120}$$

Formula (120) coincide with formula (31) if $\alpha_1 = 0$.

Formulas (109) and (114) give the next relations:

$$\frac{y}{z} = \frac{M_{2m}}{V_3 + M_{3V}} = \frac{M_{3m}}{r(1-\alpha_3)(C_3 t + V_3)} \tag{121}$$

Consistency condition for the system (SI):

$$\frac{M_{1m}}{M_{2m}} = \frac{C_3(V_1 + M_{1V})}{C_2(V_3 + M_{3V})} \tag{122}$$

Equations (110) and (111) lead to the following identities:

$$m = \frac{C_2 + C_3 - V_1}{V_1} \tag{123}$$

$$M_V = C_2 + V_2(1+m) - V \tag{124}$$

$$M = mV \tag{125}$$

$$M_m = M - M_V \tag{126}$$

Relations (119) – (126) are follow from systems (SI) – (SIII).
The following relations follow from transformational rules (SIV):

$$y = \frac{C + V}{Ct + V} \tag{127}$$

$$z = \frac{M - yM_V}{M_m} = \frac{C_3 ty}{M_{1m}} \tag{128}$$

We have also the next relations (follow from (108), (113) and (114)):

$$M_{1m} = \frac{C_3 ty}{z} = \frac{(1-\alpha_1) r (C_1 x + V_1 y)}{z} \tag{129}$$

$$M_{1V} = \alpha_1 r (C_1 t + V_1) \tag{130}$$

Parameters $\alpha_1; \alpha_2; \alpha_3$ satisfy the following conditions which can be deduced after some computations.

1) Parameter α_1 is smallest root of the following square equation:

$$\left[P(C_2+C_3)+V_1C_1C_3\right]\alpha_1^2+\left[V_1C_3(C_2-C_1)-PC_2-Q(C_2+C_3)\right]\alpha_1+(QC_2-V_1C_2C_3)=0 \quad (131)$$

$$P=V_1C_3+C(M_V+V_3) \quad (132)$$

$$Q=(M_V+V_3)(C_1+C_2)-V_2C_3 \quad (133)$$

2) Parameter α_2 is arbitrary number in interval (0;1).

$0<\alpha_2<1$ - is arbitrary number. (134)

We can choose this parameter so that to obtain the realistic solution.

3) Parameter α_3 satisfies to relation:

$$\alpha_3=(1-\alpha_2)\cdot\left(\frac{C_2x+V_2y}{C_3x+V_3y}\right)-\frac{yV_3}{r(C_3x+V_3y)} \quad (135)$$

Value V can be found if organic composition k of the capital is known:

$$V=kC=\frac{rC}{m-r} \quad (136)$$

The following square equation for parameter k can be obtained after very long computations (it is necessary to substitute (136) and (124) into (131)):

$$Ak^2+Bk+E=0 \quad (137)$$

$$A=A_1X^2+mC(XB_1+mCD_1) \quad (138)$$

$$B=-(2C_3XA_1+mCC_3B_1) \quad (139)$$

$$E=A_1C_3^2 \quad (140)$$

$$A_1=V_1C_1C_3+(C_2+C_3)\left[C(C_2+mV_2)-V_1(C_1+C_2)\right] \quad (141)$$

$$B_1=V_1C_3(C_2-C_1)-C_2\left[C(C_2+mV_2)-V_1(C_1+C_2)\right]-$$
$$-(C_2+C_3)\left[V_2(C_1+C_2)(1+m)+(C_2-V_1-V_2)(C_1+C_2)-V_2C_3\right] \quad (142)$$

$$D_1=C_2\left[V_2(C_1+C_2)(1+m)+(C_2-V_1-V_2)(C_1+C_2)-V_2C_3\right]-V_1C_2C_3 \quad (143)$$

$$X=m(C_1+C_2)-C_3 \quad (144)$$

Value V_3 can be calculated as follows: $V_3=V-V_1-V_2$ (145)

SOLUTION IS COMPLETED.

Supplement V contains numerical example of transformation in the model-2.

Calleja (2010) recently published paper devoted to solution of transformation problem in Model-2. System of equations (RSD) on page 43 describe balance-conditions in "values". System of equations (RSP) on page 44 describe balance-conditions in "prices of production" and formulate Marx's "transformational rules". It gives 12 equations for 12 variables. Author illustrates solution by means of numerical examples. Emilio Calleja's equations are fulfilled in our solution but Calleja's solutions don't satisfy very important condition:

$$P_{21}=(C_{21}\pi^C+V_{21}\pi^V)(1+r)=D_{21}\pi^V$$ [10]

[10] The following designations are used in Calleja' (2010) paper: π^C and π^V - multipliers transferring "values" into "prices of production" (coefficients x and y in our paper); P_{21} - price of production in sub-department IIa, D_{21} - "value" of goods produced in sub-department IIa.

Let's compare for example Tables 13.1 (value-structure in form "labor costs") and 13.3 (for "prices of production"). We see that $D_{21} = 300 = P_{21}$ though $\pi^V = 0.806 < 1$. Price of production of goods of sub-department IIa is less than value. Consequently volume of output after transformation obviously became larger. Consequently Emilio Calleja supposes that transformation is accompanied by some change in volumes of production. This result is incompatible with problem statement of current transformation when we describe algorithm by means of which total surplus-value perpetually redistributed between capitalists whereas volumes of current production are fixed.

X. Current transformation of "values" into "prices of production" in Model-4.

Let's consider the problem of current transformation in more realistic Model-4 in which the labor of capitalists is taken into account as a part of labor embodied in value of production. Capitalists participate in production process as managers and entrepreneurs. We have cited above (p.4) passages of the "Capital" where Marx emphasizes this point. Ideological interpretation of Marxian system has led to very simplified understanding of a role of capitalist's activity in a process of production. Capitalists are not only owners obtaining the profit but they are also active participants in a process of creation of product and realization of value of this product. The labor of capitalists should be included into the embodied socially-necessary labor ("value" of production). If capitalist doesn't participate in production process then his functions should be executed by hired managing director. Salary of hired director is included in variable capital of enterprise. Labor of hired managing director is included in embodied labor i.e. value of product. This is true as well in the case when a capitalist himself carries out functions of a businessman and an operating director at own enterprise. Models-1 and 2 don't take into account this important aspect of capitalist economy. These are too simplified (crude) models which can't fully express all essential features of capitalist economy.

The impact of capitalist's labor in each department $n = 1; 2; 3$ can be presented mathematically by means of three new variables γ_n:

$$\gamma_m = \frac{V_{\text{capitalists}}}{V} = \frac{V_{\text{capitalists}}}{V_{\text{capitalists}} + V_{\text{hired workers}}} \equiv \frac{V_{\text{cap.}}}{V_{\text{cap.}} + V_{\text{work.}}} \tag{146}$$

$$V_{cap.n} = \gamma_n \cdot V \qquad n = 1; 2; 3 \tag{147}$$

Designation V here is sum of variable capital $V_{work.}$ and part $V_{cap.}$ of surplus-value. Wage of capitalists is included in surplus-value. Value $V_{cap.}$ is not included into variable capital if capitalist himself carries out functions of management. Value $V_{cap.}$ is included in variable capital if the labor of hired managing director is used at enterprise.

"Net surplus-value" (NSV) equals to "surplus-value" minus "wage of capitalists". NSV can be expended (in the economy with simple reproduction) either onto "necessities of life" or "luxury goods". Let's designate as $\delta_n \equiv 1 - \alpha_n$ the share of NSV directed onto buying of "necessities of life"[11]. Let's designate as β_n the share of NSV directed onto buying of "luxury goods". Since "surplus-value" created in a department can differ from surplus-value consumed by capitalists of

[11] Parameters $\alpha_n = 1 - \delta_n$ are more convenient in computations.

this department parameters β_n can exceed 1 whereas parameters α_n and β_n lie in interval (0;1). We have nine new variables: $\gamma_1;\gamma_2;\gamma_3;\alpha_1;\alpha_2;\alpha_3;\beta_1;\beta_2;\beta_3$.

Table 12. Value-structure in form "labor costs" for the Model-4.

	C	$V_{\text{раб.}}$	$V_{\text{кап.}}$	V	M	W
I	C_1	$(1-\gamma_1)V_1$	$\gamma_1 V_1$	V_1	mV_1	C
II	C_2	$(1-\gamma_2)V_2$	$\gamma_2 V_2$	V_2	mV_2	$V+M_V$
III	C_3	$(1-\gamma_3)V_3$	$\gamma_3 V_3$	V_3	mV_3	M_m
SUM:	C	$(1-\gamma)V$	γV	V	$M_V+M_m=mV$	

Table 13. Value-structure in form "labor commanded" for the Model-4.

	C	$V_{\text{раб.}}$	$V_{\text{кап.}}$	V	M_V	M_m	W
I	C_1	$(1-\gamma_1)V_1$	$\gamma_1 V_1$	V_1	$(1-\alpha_1)mV_1$	$\beta_1 mV_1$	W_1
II	C_2	$(1-\gamma_2)V_2$	$\gamma_2 V_2$	V_2	$(1-\alpha_2)mV_2$	$\beta_2 mV_2$	W_2
III	C_3	$(1-\gamma_3)V_3$	$\gamma_3 V_3$	V_3	$(1-\alpha_3)mV_3$	$\beta_3 mV_3$	W_3
SUM:	C	$(1-\gamma)V$	γV	V	M_V	M_m	

Table 14. Input-flows matrix in equilibrium prices for the Model-4.

	C	$V_{\text{раб.}}$	$V_{\text{кап.}}$	M_V	M_m	W
I	xC_1	$y(1-\gamma_1)V_1$	$y\gamma_1 V_1$	$y(1-\alpha_1)mV_1$	$z\beta_1 mV_1$	xC
	$K_1 = xC_1 + y(1-\gamma_1)V_1$		$P_1 = rK_1 = y\gamma_1 V_1 + y(1-\alpha_1)mV_1 + z\beta_1 mV_1$			
II	xC_2	$y(1-\gamma_2)V_2$	$y\gamma_2 V_2$	$y(1-\alpha_2)mV_2$	$z\beta_2 mV_2$	$y(V+M_V)$
	$K_2 = xC_2 + y(1-\gamma_2)V_2$		$P_2 = rK_2 = y\gamma_2 V_2 + y(1-\alpha_2)mV_2 + z\beta_2 mV_2$			
III	xC_3	$y(1-\gamma_3)V_3$	$y\gamma_3 V_3$	$y(1-\alpha_3)mV_3$	$z\beta_3 mV_3$	zM_m
	$K_3 = xC_3 + y(1-\gamma_3)V_3$		$P_3 = rK_3 = y\gamma_3 V_3 + y(1-\alpha_3)mV_3 + z\beta_3 mV_3$			
SUM:	xC	$y(1-\gamma)V$	$y\gamma V$	yM_V	zM_m	

MATHEMATICAL FORMULATION OF TRANSFORMATION PROBLEM IN MODEL-4.

We have **Tables 12 - 14** which describe value-structure in "labor costs" and "labor commanded" forms and structure of input-flows matrix in equilibrium prices. Let's introduce auxiliary variables α and γ in such way:

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = \alpha V \quad (148)$$

$$\gamma_1 V_1 + \gamma_2 V_2 + \gamma_3 V_3 = \gamma V \quad (149)$$

I. Trivial balance-conditions in "values":

$$C_2 + V_2 (1+m) = V + mV - \alpha m V \quad (150)$$

$$C_3 + V_3 (1+m) = \alpha m V \quad (151)$$

$$M_m \equiv M_{1m} + M_{2m} + M_{3m} = \alpha m V \quad (152)$$

$$M_V \equiv M_{1V} + M_{2V} + M_{3V} = (1-\alpha) m V \quad (153)$$

II. Conditions of unchanging structure of capitalists' consumption:

$$r\left[xC_1 + y(1-\gamma_1)V_1\right] = yM_{1V} + zM_{1m} \quad (154)$$

$$r\left[xC_2 + y(1-\gamma_2)V_2\right] = yM_{2V} + zM_{2m} \quad (155)$$

$$r\left[xC_3 + y(1-\gamma_3)V_3\right] = yM_{3V} + zM_{3m} \quad (156)$$

III. Trivial balance-conditions in "prices of production":

$$\left[xC_1 + y(1-\gamma_1)V_1\right](1+r) = xC \quad (157)$$

$$\left[xC_2 + y(1-\gamma_2)V_2\right](1+r) = y(V + mV - \alpha m V) \quad (158)$$

$$\left[xC_3 + y(1-\gamma_3)V_3\right](1+r) = z\alpha m V \quad (159)$$

IV. Marx's transformational rules:

$$\left[xC + y(1-\gamma)V\right](1+r) = C + V(1+m) \quad (160)$$

$$\left[xC + y(1-\gamma)V\right]r = mV + \gamma V \quad (161)$$

$$r = \frac{mV + \gamma V}{C + (1-\gamma)V} \quad (162)$$

V. Relations between parameters α_n and β_n:

$$M_{nV} = (1-\alpha_n) m V_n ; \quad n = 1;2;3 \quad (163)$$

$$M_{nm} = \beta_n m V_n ; \quad n = 1;2;3 \quad (164)$$

SOLUTION OF TRANSFORMATION PROBLEM IN MODEL-4[12].

Parameters $\alpha_1;\alpha_2;\gamma_1;\gamma_2$ can be chosen arbitrary.

We find from (150) – (151):

$$m = \frac{C_2 + C_3}{V_1} - 1 \tag{165}$$

$$\alpha = \frac{C_3 + V_3\left(1+m\right)}{mV} \tag{166}$$

Square equation relatively variable $t = \frac{x}{y}$ follows from (157)-(158):

$$CC_2t^2 + \left[CV_2\left(1-\gamma_2\right) - VC_1\left(1+m\left(1-\alpha\right)\right)\right]t - VV_1\left(1-\gamma_1\right)\left(1+m\left(1-\alpha\right)\right) = 0 \tag{167}$$

Relation (162) follows from (160)-(161):

$$r = \frac{V\left(m+\gamma\right)}{C + V\left(1-\gamma\right)} \tag{162}$$

Let's substitute (162) into (158). We find after some transformations:

$$\gamma = 1 - \frac{\left[C_2t + V_2\left(1-\gamma_2\right)\right]\cdot\left[C + V\left(1+m\right)\right] - CV\left[1+m\left(1-\alpha\right)\right]}{\left[1+m\left(1-\alpha\right)\right]V^2} \tag{168}$$

We obtain from (157) and (160):

$$x = \frac{\left[C + V\left(1+m\right)\right]\left[C_1t + V_1\left(1-\gamma_1\right)\right]}{C\left[Ct + V\left(1-\gamma\right)\right]}; \qquad y = \frac{x}{t} \tag{169}$$

We find from (159):

$$z = \frac{\left[xC_3 + y\left(1-\gamma_3\right)V_3\right]\left(1+r\right)}{\alpha mV} \tag{170}$$

Parameters α_3 and γ_3 can be calculated from relations (148)-(149).

$$\alpha_3 = \frac{\alpha V - \alpha_1V_1 - \alpha_2V_2}{V_3} \tag{171}$$

$$\gamma_3 = \frac{\gamma V - \gamma_1V_1 - \gamma_2V_2}{V_3} \tag{172}$$

Parameters β_n are computed by means of substitution of formulas (163)-(164) into (154)-(156).

$$\beta_n = \frac{r\left[xC_1 + y\left(1-\gamma_1\right)V_1\right] - y\left(1-\alpha_n\right)mV_n}{zmV_n}; \quad n = 1;2;3 \tag{172}$$

SOLUTION IS COMPLETED.

The numerical example of transformation in Model-4 is presented in **Table 15.** We see that transformation of "values" into "prices of production" is possible even if non-trivial balance-conditions are violated.

Non-trivial balance-conditions are fulfilled if the following relations between parameters take place: $\alpha_1 = \alpha$ and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$. **Table 16** illustrates this conclusion by means of numerical example. The proof non-trivial balance-conditions in Model-4 under relations $\alpha_1 = \alpha$ and

[12] See worksheet "Mod'-4" in technical Excel-file with results of calculations on the base of formulas (165) – (172).

$\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ is given in **Supplement VI.** We used program "Mathematica 5.2" in order to prove non-trivial balance-condition:

$$xC_2 = y(V_1 + M_{1V}) = yV_1(1 + m(1-\alpha)) \text{; or} \quad (173)$$

$$tC_2 = V_1(1 + m(1-\alpha)) \quad (174)$$

Other non-trivial balance-conditions follow from (173) and trivial balance-conditions. We took equations (165) – (168) [(E1) – (E3)] and additional condition for the difference between left and right parts in formula (174):

$$\begin{cases} CC_2t^2 + \left[CV_2(1-\gamma) - VC_1(1+m(1-\alpha))\right]t - VV_1(1-\gamma)(1+m(1-\alpha)) = 0; \text{ since } \gamma_1 = \gamma_2 = \gamma & \text{(E1)} \\ \gamma = 1 - \dfrac{\left[C_2t + V_2(1-\gamma)\right]\cdot\left[C + V(1+m)\right] - CV\left[1+m(1-\alpha)\right]}{\left[1+m(1-\alpha)\right]V^2} & \text{(E2)} \\ m = \dfrac{C_2 + C_3}{V_1} - 1 & \text{(E3)} \\ \alpha = \dfrac{C_3 + V_3(1+m)}{mV} & \text{(E4)} \\ h = t\cdot C_2 - V_1\left[1 + (1-\alpha)m\right]; \text{ since } \alpha_1 = \alpha & \text{(E5)} \end{cases}$$

If $h = 0$ then it means that non-trivial balance conditions are carried out. Calculations in "Mathematica 5.2" demonstrate that it is valid. Consequently non-trivial balance conditions indeed are executed in Model-4 if interrelations $\alpha_1 = \alpha$ and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ are carried out. We argued in **Section VI** that interrelations $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ were carried out in early capitalist economy owing to high mobility of merchant capital which could be easily transferred from one department into another department. Therefore capitals of separate merchants were dispersed between all three departments. Capital in each department consisted of many different capitals with different values α and γ. The "Law of big numbers" guaranteed the equalizing of these parameters between departments: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$; $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$. Therefore non-trivial balance-conditions were executed in early capitalist economy.

Solution of "transformation problem" in Model-4 exists under any choice of value-matrix: $(C_1; C_2; C_3; V_1; V_2; V_3)$ (worksheet "Mod'-4" in technical Excel-file). Solutions satisfy to non-trivial balance-conditions if equalities $\alpha_1 = \alpha$ and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ are carried out. We may change coefficients in value-matrix observing how it influences on solution. Only solutions with positive and admissible values of all variables should be chosen as realistic solutions. Non-realistic solutions are possible in Model-4. **Supplement VII** (and worksheet "Mod-4-nonrealistic" in technical Excel-file) contains numerical example of non-realistic solution.

Table 15. Transformation of “values” into “prices of production” in Model-4 without execution of non-trivial balance-conditions.

	Value of products consumed in each department (labor commanded)								
	C	V work.	V cap.	V	Mv	Mm	M	m	W"
I	2000	574	126	700	155	3017	3172	4.53	5872
II	1000	320	80	400	35	1572	1608	4.02	3008
III	2800	831	169	1000	881	3639	4520	4.52	8320
SUM:	**5800**	1725	375	2100	1071	**8229**	9300		**17200**
	CAPITAL (K) =	**7525.48**							
r = (M + V cap.) : (C + V work.) =			**1.286**	V + Mv =	**3171**		P = M + V cap. =		**9675**

MODEL - 4. Non-trivial balance-conditions may be violated.

	Value of products produced in departments (labor costs)							
	C	V work.	V cap.	V	M	m	W	
I	2000	574	126	700	3100	4.43	**5800**	W1' = W 1*x
II	1000	320	80	400	1771	4.43	**3171**	W2' = W2 *y
III	2800	831	169	1000	4429	4.43	**8229**	W3'=W3*z
SUM:	**5800**	1725	375	2100	9300	4.43	**17200**	
	CAPITAL (K) =	**7525**			P = M + V cap. =		**9675**	

	Prices of production								
	C'	V work.	V cap.	V'	M'v	M'm	P' = r * K	r'	W'
I	2029	546	120	665	147	3043	3310	1.286	**5885**
II	1015	304	76	380	34	1586	1696	1.286	**3014**
III	2841	790	160	950	837	3671	4669	1.286	**8300**
SUM:	**5885**	1640	356	1996	1018	**8300**	**9675**	**1.286**	**17200**
	CAPITAL (K) =	**7525**		V' + M'v =	**3014**			alpha =	0.885
	gamma 1 =	0.180		gamma2 =	0.200	gamma3 =	0.169	gamma =	0.178

Cn' = Cn * x	Kn = Cn + V раб.	x =	1.015
V'n раб. = Vn раб. * y	K'n = C'n + V' раб.	y =	0.950
V'n кап. = Vn кап. * y	P'n = r * K'n	z =	1.009
V' = V * y	W'n = (1 + r) * K'n	alpha1 =	0.950
M'vn = Mvn * y	V'n кап. + M'vn + M'mn = P'n	alpha2 =	0.980
M'mn = Mmn * z	r' = P' : K'	alpha3 =	0.801

Table 16. Transformation of “values” into “prices of production” in Model-4 with execution of non-trivial balance-conditions. Relations $\alpha_1 = \alpha$ and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ are carried out.

	Value of products consumed in each department (labor commanded)								
	C	V work.	V cap.	V	Mv	Mm	M	m	W''
I	2000	673	27	700	357	2824	3181	4.54	5881
II	1000	384	16	400	531	1128	1660	4.15	3060
III	2800	961	39	1000	183	4277	4460	4.46	8260
ВСЕГО:	5800	2018	82	2100	1071	8229	9300		17200
	КАПИТАЛ (K) =		7818						
r = (M + V cap.) : (C + V work.) =			1.200	V + Mv =	3171			P = M + V cap. =	9382

MODEL - 4. Non-trivial balance-conditions are executed.

	Value of products produced in departments (labor costs)							
	C	V work.	V cap.	V	M	m	W	
I	2000	673	27	700	3100	4	5800	W1' = W 1*x
II	1000	384	16	400	1771	4	3171	W2' = W2 *y
III	2800	961	39	1000	4429	4	8229	W3'=W3*z
ВСЕГО:	5800	2018	82	2100	9300	4	17200	
	КАПИТАЛ (K) =		7818		P = M + V cap. =		9382	

	Prices of production								
	C'	V work.	V cap.	V'	M'v	M'm	P' = r * K	r'	W'
I	2028	645	26	672	343	2840	3208	1.200	5882
II	1014	369	15	384	510	1135	1660	1.200	3042
III	2840	922	37	959	175	4301	4514	1.200	8275
ВСЕГО:	5882	1936	78	2015	1028	8275	9382	1.200	17200
	КАПИТАЛ (K) =		7818	V' + M'v =	3042			alpha =	0.885
	gamma 1 =	0.039		gamma2 =	0.039	gamma3 =	0.039	gamma =	0.039
	Cn' = Cn * x				Kn = Cn + V раб.			x =	1.014
	V'n раб. = Vn раб. * y				K'n = C'n + V' раб.			y =	0.959
	V'n кап. = Vn кап. * y				P'n = r * K'n			z =	1.006
	V' = V * y				W'n = (1 + r) * K'n			alpha1 =	0.885
	M'vn = Mvn * y				V'n кап. + M'vn + M'mn = P'n			alpha2 =	0.700
	M'mn = Mmn * z				r' = P' : K'			alpha3 =	0.959

XI. NTBC-invariant conversions: Model-4 → Model-2 → Model-1.

Model-4 with $\alpha_1 = \alpha$ and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ in which non-trivial balance-conditions are executed can be converted into Model-2 in which non-trivial balance-conditions will be carried out also.

A. NTBC-INVARIANT CONVERSION: Model-4 → Model-2.

Model-4 can be presented as Model-2 by means of introduction of "effective rate of surplus value":

$$m_{ef} = \frac{V_{\text{кап.}} + mV}{V_{\text{раб.}}} \qquad (175)$$

Model-2 doesn't take into account a labor of capitalists. It means that capitalists' labor is included into embodied labor ("value" of product) as a labor of hired workers. Therefore effective rate of surplus value in Model-2 is higher than in Model-4.

Table 17. NTBC-invariant conversion: Model-4 (Table 16) → Model-2.

Transformation of model-4 into model-2

MODEL-2

m ef. =	4.65

Labor costs

	C	V work. = V	Mv	M	m	W
I	2000	673	384	3127	4.65	**5800**
II	1000	384	547	1787	4.65	3171
III	2800	961	222	4468	4.65	8229
ВСЕГО:	**5800**	2018	1153	**9382**	4.65	**17200**
		V work. + Mv =	3171			

МОДЕЛЬ-2

Labor commanded

	C	V work. = V	Mv	Mm	M	m	W
I	2000	673	384	2824	3208	4.77	5881
II	1000	384	547	1128	1675	4.36	3060
III	2800	961	222	4277	4499	4.68	8260
ВСЕГО:	**5800**	2018	1153	8229	**9382**	4.65	**17200**
		V work. + Mv =	3171				

МОДЕЛЬ-2

Production prices

	C	V work. = V	Mv	Mm	P	r	W
I	2028	**645**	**369**	**2840**	3208	1.200	**5882**
II	**1014**	369	525	**1135**	1660	1.200	**3042**
III	**2840**	**922**	**213**	4301	4514	1.200	**8275**
ВСЕГО:	**5882**	1936	1106	**8275**	**9382**	1.200	**17200**
		V work. + Mv =	**3042**				

Table 17 illustrates NTBC-invariant conversion of Model-4 into Model-2 for numerical example of **Table 16.** Technical Excel-file contains the program of calculations (worksheet "Mod-4").

B. NTBC-INVARIANT CONVERSIONS: Model-2 → new Model-2 → Model-1.

Any Model-2 in which non-trivial balance-conditions are carried out can be presented as other (new) Model-2 by means of redefinition of sub-departments IIa and IIb. Let's divide production of sub-department IIa into two parts: 1) production of goods for "workers" and 2) production of goods for "capitalists". We can include into new sub-department II'b part of branches of sub-department IIa from the second group (production of goods for capitalists). It gives us new Model-2. Conversion into Model-1 is based on inclusion of all branches producing goods for capitalists into new sub-department II'b. Model-1 as well as Model-2 satisfies to non-trivial balance-conditions if conversion is based on algorithm presented in Table 18 and formulas (176) – (185).

Table 18. NTBC-invariant conversion: Model-2 → Model-1.

INITIAL MODEL-2.

I. $C_1 + V_1 + (M_{1V} + M_{1m})$ - means of production

IIa. $C_2 + V_2 + (M_{2V} + M_{2m})$ - necessities of life

IIb. $C_3 + V_3 + (M_{3V} + M_{3m})$ - luxury goods

DIVISION OF SUB-DEPARTMENT IIa

$$\text{I.}\quad C_1 + V_1 + \left(\underbrace{\gamma_1 M_{1V}}_{M'_{1V}} + \underbrace{(1-\gamma_1) M_{1V} + M_{1m}}_{M'_{1m}} \right)$$

$$\text{IIa-1.}\quad aC_2 + \beta V_2 + \left(\underbrace{\mu_1 \gamma_2 M_{2V}}_{M'_{2V} \equiv M_{IIa-1V}} + \underbrace{\mu_2 (1-\gamma_2) M_{2V} + \mu_3 M_{2m}}_{M'_{2m} \equiv M_{IIa-1m}} \right)$$

$$\text{IIa-2.}\quad (1-a) C_2 + (1-\beta) V_2 + \left(\underbrace{(1-\mu_1) \gamma_2 M_{2V}}_{M_{IIa-2V}} + \underbrace{(1-\mu_2)(1-\gamma_2) M_{2V} + (1-\mu_3) M_{2m}}_{M_{IIa-2m}} \right)$$

$$\text{IIb.}\quad C_3 + V_3 + \left(\underbrace{\gamma_3 M_{3V}}_{M'_{IIb}} + (1-\gamma_3) M_{3V} + M_{3m} \right)$$

NEW MODEL - 2.

I. $C_1 + (V_1 + M'_{1V}) + (M'_{1m})$

II'a. $aC_2 + \beta V_2 + M'_{2V} + M'_{2m}$

II'b. $(C_3 + (1-a) C_2) + (V_3 + (1-\beta) V_2 + M'_{3V}) + M'_{3m}$

Algorithm of conversion is described by means of the following formulas:

$$M'_{2V} = \mu_1 \gamma_2 M_{2V} \tag{176}$$

$$M'_{2m} = \mu_2 \left(1-\gamma_2\right) M_{2V} + \mu_3 M_{3m} \tag{177}$$

$$M'_{3V} = \gamma_3 M_{3V} + \left(1-\mu_1\right) \gamma_2 M_{2V} \tag{178}$$

$$M'_{3m} = M_{3m} + \left(1-\gamma_3\right) M_{3V} + \left(1-\mu_2\right)\left(1-\gamma_2\right) M_{2V} + \left(1-\mu_3\right) M_{2m} \tag{179}$$

$$a = \frac{V_1 + \gamma_1 M_{1V}}{C_2} \tag{180}$$

$$\mu_3 = \frac{V_3 + \left(1-\beta\right) V_2 + \gamma_3 M_{3V} + M_{2V}\left[\gamma_2\left(1-\mu_1\right) - \mu_2\left(1-\gamma_2\right)\right]}{M_{2m}} \tag{181}$$

$$\gamma_3 = \frac{r\left(\alpha C_2 + \beta V_2\right) - V_3 - \left(1-\beta\right) V_2 - M_{2V}\gamma_2}{M_{3V}} \tag{182}$$

$$r = \frac{M_{1V} + M_{1m}}{C_1 + V_1} \tag{183}$$

$$M_m = \sum_{n=1}^{3} M_{nm}\,; \qquad M_V = \sum_{n=1}^{3} M_{nV} \tag{184}$$

Parameters $a;\beta;\gamma_n;\mu_n$ ($n = 1;2;3$) lie within interval (0;1). Parameters $\gamma_1;\gamma_2;\mu_1;\mu_2;\beta$ are arbitrary numbers which lie in admissible limits. Algorithm based on formulas (176) – (184) gives conversion of Model-2 into Model-1 if equality $\gamma_1 = \gamma_2 = \gamma_3 = 0$ is fulfilled. We have in this partial case:

$$\beta = \frac{V_2 + V_3 - r\alpha C_2}{V_2\left(1+r\right)} \tag{185}$$

Tables 19-22 illustrate conversions: Model-2 → new Model-2 → Model-1 for Marx's numerical example and in general case.

Technical Excel-file (worksheets "Mod-2-2"; "Mod'-2-2"; "Mod-2-1" and "Mod'-2-1") contains programs of conversion: Model-2 → new Model-2 → Model-1.

We may convert Model-4 into Model-2 and further into Model-1 by means of NTBC-invariant conversions. Consequently Model-1 obtained from Model-4 by means of NTBC-invariant algorithm satisfies non-trivial balance-conditions. All results of previous Sections of our paper concerning Model-1 in which NTBC are carried out can be used for Model-1 constructed from realistic Model-4 by means of NTBC-invariant algorithm of conversion.

If NTBC were carried out in Model-4 of early capitalist economy then NTBC were executed both in Model-1 and Model-2 obtained by means of NTBC-conversion from Model-4.

Table 19. Conversion Model-2 → Model-2 for Marx's numerical example.

INITIAL MODEL-2 (Marx's numerical example).							
	C	V	Mv	Mm	C:V	r	W
I	4000	1000	600	400	4.00	0.20	6000
II-a	1600	400	240	160	4.00	0.20	2400
II-b	400	100	60	40	4.00	0.20	600
	6000	1500	900	600	4.00	0.20	9000
DIVISION OF PRODUCTION II-a:							
	C'	V'	M'v	M'm	C:V	r	W'
I	4000	1000	420	580	4.00	0.20	6000
II-a-1	1420	360	183.6	172.4	3.94	0.20	2136
II-a-2	180	40	20.4	23.6	4.50	0.20	264
II-b	400	100	12	88	4.00	0.20	600
	6000	1500	636	864	4.00	0.20	9000
NEW MODEL-2.							
	C"	V"	M"v	M"m	C:V	r	W"
I	4000	1000	420	580	4.00	0.20	6000
II-a	1420	360	183.6	172.4	3.94	0.20	2136
II-b	580	140	32.4	111.6	4.14	0.20	864
	6000	1500	636	864	4.00	0.20	9000

gamma1	0.7		mu1	0.9	alpha	0.8875
gamma2	0.85		mu2	0.7	beta	0.9
gamma3	0.2		mu3	0.92	r	0.2

Table 20. Conversion Model-2 → Model-1 for Marx's numerical example.

INITIAL MODEL-2 (Marx's numerical example).							
V	C	V	Mv	Mm	C:V	r	W
I	4000	1000	600	400	4.00	0.20	6000
II-a	1600	400	240	160	4.00	0.20	2400
II-b	400	100	60	40	4.00	0.20	600
	6000	1500	900	600	4.00	0.20	9000
DIVISION OF PRODUCTION II-a:							
	C'	V'	M'v	M'm	C:V	r	W'
I	4000	1000	0	1000	4.00	0.20	6000
II-a-1	1000	250	0	250	4.00	0.20	1500
II-a-2	600	150	0	150	4.00	0.20	900
II-b	400	100	0	100	4.00	0.20	600
	6000	1500	0	1500	4.00	0.20	9000
NEW MODEL-2.							
	C"	V"	M"v	M"m	C:V	r	W"
I	4000	1000	0	1000	4.00	0.20	6000
II-a	1000	250	0	250	4.00	0.20	1500
II-b	1000	250	0	250	4.00	0.20	1500
	6000	1500	0	1500	4.00	0.20	9000

gamma1	0		mu1	0.8	alpha	0.625
gamma2	0		mu2	0.7	beta	0.625
gamma3	0		mu3	0.5125	r	0.2

Table 21. Conversion Model-2 → Model-2 in general case.

INITIAL MODEL-2							
	C	V	Mv	Mm	C:V	r	W
I	4000	1000	300	700	4.00	0.20	6000
II-a	1300	900	280	160	1.44	0.20	2640
II-b	700	100	60	100	7.00	0.20	960
	6000	2000	640	960	3.00	0.20	9600
DIVISION OF PRODUCTION II-a:							
	C'	V'	M'v	M'm	C:V	r	W'
I	4000	1000	240	760	4.00	0.20	6000
II-a-1	1240	720	50.4	341.6	1.72	0.20	2352
II-a-2	60	180	5.6	42.4	0.33	0.20	288
II-b	700	100	56	104	7.00	0.20	960
	6000	2000	352	1248	3.00	0.20	9600
NEW MODEL-2.							
	C"	V"	M"v	M"m	C:V	r	W"
I	4000	1000	240	760	4.00	0.20	6000
II-a	1240	720	50.4	341.6	1.72	0.20	2352
II-b	760	280	61.6	146.4	2.71	0.20	1248
	6000	2000	352	1248	3.00	0.20	9600

gamma1	0.8		mu1	0.9	alpha	0.9538462
gamma2	0.2		mu2	0.85	beta	0.8
gamma3	0.9333333		mu3	0.945	r	0.2

Table 22. Conversion: Model-2 → Model-1 in general case.

INITIAL MODEL-2.							
	C	V	Mv	Mm	C:V	r	W
I	4000	1000	300	700	4.00	0.20	6000
II-a	1300	900	280	160	1.44	0.20	2640
II-b	700	100	60	100	7.00	0.20	960
	6000	2000	640	960	3.00	0.20	9600
DIVISION OF PRODUCTION II-a:							
	C'	V'	M'v	M'm	C:V	r	W'
I	4000.00	1000.00	0.000	1000.00	4.00	0.20	6000
II-a-1	1000.00	666.67	0.000	333.33	1.50	0.20	2000
II-a-2	300.00	233.33	0.000	106.67	1.29	0.20	640
II-b	700.00	100.00	0.000	160.00	7.00	0.20	960
	6000	2000	0.000	1600	3.00	0.20	9600
MODEL-1.							
	C"	V"	M"v	M"m	C:V	r	W"
I	4000	1000.00	0.000	1000.00	4.00	0.20	6000
II-a	1000	666.67	0.000	333.33	1.50	0.20	2000
II-b	1000	333.33	0.000	266.67	3.00	0.20	1600
	6000	2000	0.000	1600	3.00	0.20	9600

gamma1	0		mu1	0.8	alpha	0.769
gamma2	0		mu2	0.7	beta	0.741
gamma3	0.00		mu3	0.858	r	0.2

XII. Conclusion.

This paper is devoted to the dissolution of transformation problem in capitalist economy with simple reproduction. We considered three models: Models-1 and 2 in which "labor of capitalists" does not take into account and more realistic Model-4 in which this factor accounted. Realistic solution exist both in Model-1 and in Model-2 if "non-trivial balance conditions" (NTBC) are carried out. Marx has introduced NTBC in chapter XX where he considered a numerical example of capitalist economy with simple reproduction. NTBC most likely were carried out in early capitalist economy in which the merchant capital prevailed. High mobility of merchant capital stimulated the dispersal of separate capitals between industries of all three departments. It was leading to equalizing in the Model-4 of parameters $\delta \equiv 1-\alpha$ (percentage of capitalists' expenditures onto "necessities of life" in net surplus-value) and γ (percentage of "capitalists' wage" in full payment of labor) between departments ("law of big numbers"): $\alpha_1 = \alpha_2 = \alpha_3$; $\gamma_1 = \gamma_2 = \gamma_3$. These equalities are sufficient conditions for execution of NTBC in Model-4.

Model-4 is most realistic model of capitalist simple production. Model-4 satisfying NTBC can be converted into Model-2 and Model-2 further can be converted into Model-1. These models are bound by NTBC-invariant conversions which conserve "non-trivial balance-conditions" in all three models.

We obtained solution of transformation problem in all three models. Solution of transformation problem in Models-1 and 2 exists if NTBC are carried out. Solution in Model-4 exists without imposition of NTBC. We have argued why NTBC should be fulfilled in early capitalist economy (owing to dispersal of merchant capitals between three departments of social production and owing to action of the "law of big numbers"). Therefore the transformation problem statement for early capitalist economy should include imposition of NTBC. For the same reason the problem statement of historical transformation also assumes imposing of non-trivial balance-conditions. We considered two possible ways of historical transformation. This process (transformation of exchange based on "values" into exchange based on "production prices") could occur only under condition of change in technologies and rate of a surplus value.

"Transformation problem" came about from Bortkiewicz' (1907) paper in which he stated the question whether Marx's transformation rules can be executed in Model-1. He answered - "no". Our analysis demonstrates that it is not so. Solution in Model-1 and Model-2 exists if Marx's "non-trivial balance-conditions" are taken into account. Bortkiewicz has not taken into account non-trivial balance-conditions which should be carried out in early capitalist economy with simple reproduction. As consequence he obtained wider set of solutions. Only those solutions which don't satisfy to Marx's "non-trivial conditions of the balance" (NTBC) don't satisfy also to Marx's rules of transformation.

References.

1. Abraham-Frois, G. (1979). Theory of value, prices and accumulation. A mathematical integration of Marx, von Neumann and Sraffa. London – New York: Cambridge University Press.
2. Banaji, J. (2003). Islam, the Mediterranean and the Rise of Capitalism. Paper presented to the Conference on 'Theory as history: Ernest Mandel's Historical Analysis of World Capitalism', Amsterdam, 10-11th November, 2003.
3. Bortkiewicz, L. von (1907a). Essai de rectification de la construction theorique fondamentale de Marx dans le livre 3 du Capital, in french, cahiers de l'ISEA, serie S, n.1 (19-36), or in english IEP n.2, 1952. or Bortkiewicz, L. von "On the Correction of Marx's Fundamental Theoretical Construction in the Third Volume of Capital" by Ladislaus von Bortkiewicz. Originally published in Jahrbucher fur Nationalokonomie und Statistik; July 1907; Translated by Paul M. Sweezy.
4. Bortkiewicz, L. von (1907b). Wertechnung und Preisrechnung im Marxschen System. Archiv fur Sozialwissenschaft und Sozialpolitik 25: 10–51, 445–xxx [Value and Price in the Marxian System.International Economic Papers 2 (1952): 5–60]
5. Calleja, E. D. (2010) Teoría del capital: Una aproximación matemática al esquema de reproducción de Marx. Internet Resource: http://personal.us.es/emidiaz/index_files/TeoriadelCapital.pdf
6. Dumenil, G. (1980). De la valeur aux prix de production. Paris: Economica.
7. Dumenil, G. (1983-84). Beyond the transformation riddle: a labor theory of value. Science and Society 47 (4), Winter (427-50).
8. Fine, B., Saad-Filho, A., Lapavitsas, C. (2004) Transforming the Transformation Problem: Why the 'New Interpretation' is a Wrong Turning. *Review of Radical Political Economics* 36 (1), (3-19).
9. Foley, D. (1982). The value of money, the value of labor power and the Marxian transformation problem. Review of Radical Political Economics 14(2), Summer (37-47).
10. Freeman, A. (1996). The psychopathology of Walrasian Marxism. In "Marx and Non-Equilibrium Economics", Freeman, A. and Carchedi, G. (eds) (1996), (1-29). Cheltenham: Edward Elgar.
11. Heaton, H. (1920). The Yorkshire Wollen and Worsted Industries. Oxford; p.299 and p.388.
12. Itoh, M. (1980). Value and Crisis. London: Pluto Press.
13. John, A. (1962). "Aspects of English Economic Growth in the first half of the XVIII century". In: Essays in Economic History; v. II. Ed. By E.M. Carus – Wilson. L.; 1962.
14. Jonathan, M.J. (2006). The Capitalist Spirit Unfettered: Honor, the Doctrine of Providence and Late Medieval Italian Merchants' Profit-Seeking Behavior. Baccalaureate Thesis Submitted to the Faculty of the Department of History of Vanderbilt University.
15. Kliman, A.J.. and McGlone, T. (1999). A Temporal Single-system Interpretation of Marx's Value Theory. Review of Political Economy 11(1), (33-59).
16. Kliman, A.J. (2007). Reclaiming Marx's "Capital". A Refutation of the Myth of Inconsistency. Lexington Books, a division of Rowman & Littlefield. Part of Lexington's *Raya Dunayevskaya Series in Marxism and Humanism.*
17. Laibman, D. (2004). Rhetoric and Substance in Value Theory: An appraisal of the new orthodox Marxism. In "The New Value Controversy and the Foundations of Economics", Freeman, A., Kliman, A. and Wells, J. (eds), Cheltenham, UK: Edward Elgar.
18. Lipietz, A. (1982). The 'so-called transformation problem' revisited. J. of Economic Theory 26 February
19. Mariolis, T. (2006). *A Critique of the 'New Approach' to the Transformation Problem and a Proposal.* Published in: Indian Development Review: An International Journal of Development Economics 4 (1), June, (23-37).
20. Medio, A. (1972). Profits and surplus value: appearance and reality in capitalist production. In "A Critique of Economic Theory", Huntand, E.K. and Schwartz, J. (eds.), New York: Penguin.
21. Meek, R.L. (1956). Studies in the Labour Theory of Value (London: Lawrence & Wishart).
22. Mielants, E.H. (2007). The Origins of Capitalism and the "Rise of the West". Temple University Press.
23. Morishima,M. (1973). Marxian Economics. Oxford University Press.

24. Morishima, M. and Caterhores, G. (1975). The transformation problem: a Markov process. In "Value Exploitation and Growth – Marx in the Light of Modern Economic Theory", Morishima, M (ed.), New York: McGraw – Hill.
25. Moseley, F. (1999). The 'new solution' to the transformation problem: a sympathetic critique. Review of Radical Political Economics 32 (2), (282-316).
26. Park, Cheol-Soo, (2009). The Temporal Single System: A retrospective critique. Marxism 21.
27. Roberto V. (2004). The Temporal Single-System Interpretation of Marx's Economics: A critical evaluation," Metroeconomica 55(1).
28. Roemer, J.E., *(*1980). A General Equilibrium Approach to Marxian Economics. Econometrica 48(2), (505-30).
29. O'Donnell, R. (1984). Adam Smith's labour command measure of value. UCD Centre for Economic Research Working Paper Series; No. 21.
30. Samuelson, P.A. (1971). Understanding the marxian notion of exploitation: a summary of the so-called transformation problem between Marxian values and competitive prices. Journal of Economic Literature 9(2), June (399-431).
31. Shaikh, A. (1977). Marx's theory of value and the 'transformational problem'. In "The Subtle Anatomy of Capitalism", Sohwartz, J. (ed.), Santa Monica: Goodyear.
32. Shaikh, A. (1984). The transformation from Marx to Sraffa. In "Ricardo, Marx, Sraffa", Mandell, E. (ed.), London: Verse.
33. Smith, A.: [1776] 1994, The Wealth of Nations, The Modern Library, New York. First published in 1776.
34. Steedman, I. (1977). Marx after Sraffa. London: New Left Books.
35. Sweezy, P.M. (1942). The Theory of Capitalist Development. New York: Monthly Review Press, ch.7.
36. Sweezy, P. M. (ed) (1949). Karl Marx and the Close of His System by Eugen Bohn-Bawerk and Bohn-Bawerk's Critism of Marx by Rudolf Hilferding, N.Y.: Clifton, A.M. Kelly.
37. Wright*,* I. (2007). Nonstandard labour values. Open Discussion Papers in Economics 69.

SUPPLEMENT I. The complete list of models of capitalist economy with simple production.

MODEL	Labor of capitalists enters into value of product	Part of profit is compensation capitalists as employees	Compensation (“wage of capitalists” as employees) is expended upon:	Net profit (profit minus “wage of capitalists”) is expended on:	Workers buy:
Model-1	No	No	No compensation	“luxury goods”	“necessities of life”
Model-2	No	No	No compensation	“luxury goods” and “necessities of life”	“necessities of life”
Model-3	Yes	Yes	“necessities of life”	“luxury goods”	“necessities of life”
Model-4	Yes	Yes	“necessities of life”	“luxury goods” and “necessities of life”	“necessities of life”
Model-5	No	No	No compensation	“luxury goods”	“necessities of life” and ”luxury goods”
Model-6	No	No	No compensation	“luxury goods” and “necessities of life”	“necessities of life” and “luxury goods”
Model-7	Yes	Yes	“necessities of life”	“luxury goods”	“necessities of life” and “luxury goods”
Model-8	Yes	Yes	“necessities of life”	“luxury goods” and “necessities of life”	“necessities of life” and “luxury goods”
Model-9	Yes	Yes	“necessities of life” and “luxury goods”	“luxury goods”	“necessities of life” and “luxury goods”
Model-10	Yes	Yes	“necessities of life” and “luxury goods”	“luxury goods” and “necessities of life”	“necessities of life” and “luxury goods”

SUPPLEMENT II. Numerical examples of value-structure.

Example 1. Value-structure (SV) of exchange based on "values".

	C	V	M	m	W
I	280.00	420.00	300.00	0.71	1000.00
II	420.00	105.00	75.00	0.71	600.00
III	300.00	75.00	53.57	0.71	428.57
Σ:	1000.00	600.00	**428.57**	0.71	**2028.57**

The following parameters were used: $a = 0.4$; $b = 0.3$; $k = 0.6$.

Example 2. Value-structure of exchange in which "values" and "prices of production" coincide.

Value of products consumed in each department (labor commanded):					
	C (means of production)	V (consumer goods)	M (luxury goods)	m	W
I	420.00	280.00	300.00	1.071	1000.00
II	280.00	186.67	200.00	1.071	666.67
III	300.00	200.00	214.29	1.071	714.29
SUM:	1000.00	666.67	**714.29**	1.071	**2380.95**
Value of product produced in departments (labor cost):					
	C (transferring value)	V (necessary labor)	M (surplus labor)	m	W
I	420.00	280.00	300.00	1.071	1000.00
II	280.00	186.67	200.00	1.071	666.67
III	300.00	200.00	214.29	1.071	**714.29**
SUM:	1000.00	666.67	**714.29**	1.071	**2380.95**
Prices of production:					
	C (means of production)	V (consumer goods)	M (luxury goods)	r	W
I	420.00	280.00	300.00	0.429	1000.00
II	280.00	186.67	200.00	0.429	666.67
III	300.00	200.00	214.29	0.429	**714.29**
SUM:	1000.00	666.67	**714.29**	0.429	**2380.95**

The same color corresponds to the equal quantities.

The following parameters were used: $a = 0.6$; $b = 0.3$. Calculations give the following values: $k = 0.667$; $r = 0.429$.

Example 3. Value-structure (SP) of exchange based on “prices of production”.

Value of products consumed in each department (labor commanded):					
	C (means of production)	V (consumer goods)	M (luxury goods)	m	W
I	420.00	338.33	334.62	0.989	1092.95
II	280.00	711.67	415.38	0.584	1407.05
III	300.00	450.00	321.43	0.714	1071.43
SUM:	1000.00	1500.00	**1071.43**	0.714	**3571.43**

Value of product produced in departments (labor cost):					
	C (transferring value)	V (necessary labor)	M (surplus labor)	m	W
I	420.00	338.33	241.67	0.714	1000.00
II	280.00	711.67	508.33	0.714	1500.00
III	300.00	450.00	321.43	0.714	**1071.43**
SUM:	1000.00	1500.00	**1071.43**	0.714	**3571.43**

Prices of production:					
	C (means of production)	V (consumer goods)	M (luxury goods)	r	W
I	468.46	312.31	334.62	0.429	1115.38
II	312.31	656.92	415.38	0.429	1384.62
III	334.62	415.38	321.43	0.429	**1071.43**
SUM:	1115.38	1384.62	**1071.43**	0.429	**3571.43**

The following parameters were used: $a = 0.6$, $b = 0.3$, $k = 1.5$.
Calculations give: $x = 1.1154$, $y = 0.9231$, $z = 1$, $r = 0.429$.

SUPPLEMENT III. Numerical modelling of random distribution of capitals in model-4.

	Value-structure in form "labor commanded":								
	C	V work.	V cap.	V	Mv	Mm	M	m	W"
I	**244800**	82374	2034	**84409**	50309	377646	427954	5.07	757163
II	**128720**	49498	1209	**50707**	30433	202389	232822	4.59	412249
III	**372944**	128136	3164	**131301**	78548	576759	655307	4.99	1159552
ВСЕГО:	**746465**	260009	6407	266416	159289	**1156793**	1316083		**2328964**
	CAPITAL (K) =	**1006474**							
r = (M + V кап.) : (C + V раб.) =			**1.314**	V + Mv =	**425705.425**			P = M + V cap. =	**1322490**

Model-4. Random distribution of capitals.

	Value-structure in form "labor cost".							
	C	V work.	V cap.	V	M	m	W	
I	244800	82374	2034	84409	417256	4.94	**746465**	W1' = W 1*x
II	128720	49498	1209	50707	250658	4.94	**430086**	W2' = W2 *y
III	372944	128136	3164	131301	649058	4.94	**1153303**	W3'=W3*z
ВСЕГО:	**746465**	260009	6407	266416	1316973	4.94	**2329854**	
	CAPITAL (K) =	**1006474**			P = M + V кап. =		**1323380**	
Deviation, % =			**0.41%**		**Deviation, %**	**1.20%**	**Deviation, %**	**1.29%**
	Prices of production.				V 1+ M1v =	**129044.82**	V 3+ M3v =	**201013**

	C'	V work.	V cap.	V'	M'v	M'm	P' = r * K	r'	W'
I	248394	78906	1948	**80855**	**48190**	**379986**	430125	1.314	**757426**
II	**130611**	47414	1158	48572	29151	**203643**	233952	1.314	**411977**
III	**378420**	122741	3031	**125772**	**75241**	580333	658606	1.314	**1159767**
ВСЕГО:	**757426**	249061	6137	255199	152583	**1163963**	**1322683**	**1.314**	**2329170**
	CAPITAL (K) =	**1006487**		V' + M'v =	**407781.19**			alpha =	0.8757
	gamma 1 =	0.0241		gamma2 =	0.0238	gamma3 =	0.0241	gamma =	0.0229
	Cn' = Cn * x				Kn = Cn + V раб.			x =	1.0147
	V'n раб. = Vn раб. * y				K'n = C'n + V' раб.			y =	0.9579
	V'n кап. = Vn кап. * y				P'n = r * K'n			z =	1.0062
	V' = V * y				W'n = (1 + r) * K'n			alpha1 =	0.8794
	M'vn = Mvn * y				V'n кап. + M'vn + M'mn = P'n			alpha2 =	0.8786
	M'mn = Mmn * z				r' = P' : K'			alpha3 =	0.8790

SUPPLEMENT IV. Numerical examples of modelling of historical transformation.

Let's consider numerical examples which illustrate how the exchange based on "values" could be transformed into the exchange based on "prices of production". Consider simple case when only third department is modernized. Let technological coefficient γ_3 and rate of surplus-value be varying whereas all other values are constant.

Exchange based on "values" is initial state of economy (before transformation). This initial state in our model can be given by means of the following relations:

(SIV.1) $m = m_1 = m_2 = m_3$

(SIV.2) $\gamma_2 = \gamma_3$

(SIV.3) $x = y = z = 1$

Example 1. Parameter γ_3 increases during transition period.

Let's take the following initial values of parameters: $\gamma_1 = 0.3 > \gamma_2$; $\gamma_2 = \gamma_3 = 0.1$; $m = 1$.

The following quantities can be calculated:

(SIV.4) $\beta = \dfrac{1}{1+m}$

(SIV.5) $k = \dfrac{V}{C} = \dfrac{1-\gamma_1}{\gamma_2 + m\gamma_3}$

(SIV.6) $t = \dfrac{V_1}{C_2}$

(SIV.7) $y = \dfrac{M_2}{V_3}$

(SIV.8) $x = ty$

(SIV.9) $\dfrac{m_1}{m_2} = \dfrac{k_2}{k_3} = \dfrac{\gamma_3(1-\gamma_2)}{\gamma_2(1-\gamma_3)}$

(SIV.10) $r_1 = \dfrac{m\gamma_3(1-\gamma_1)}{\gamma_2 + m\gamma_1\gamma_3}$

(SIV.11) $r_2 = r_3 = \dfrac{m(1-\gamma_3)}{1+m\gamma_3}$

Table I(SIV). Initial Matrix of Exchange based on "values".

Value of products consumed in each department (labor commanded):							
	C	V	M	m	SUM:	r	k
I	300	350	350	1.00	1000	0.54	1.17
II	350	1575	1575	1.00	3500	0.82	4.50
III	350	1575	1575	1.00	3500	0.82	4.50
SUM:	1000	3500	3500	1.00	8000	0.78	3.50

Transformation of values into production prices in our model occurs on account of modernization of third department. Modernization is possible if this process doesn't decrease the rate of profit in third department. Let's consider "threshold case" when rate of profit in third department is constant during transition period. We will change parameters stepwise. Transition period is modeled as recurrent process consisting of many sub-periods with different values of parameters. Let's designate parameters in each sub-period as $m^{(n)};\gamma_3^{(n)}$ where n is the number of sub-period. The rate of profit is the same in each sub-period n if rate of surplus-value satisfy to the following relation:

(SIV.12) $$m^{(n)} = \frac{r_3}{1-\gamma_3^{(n)}\left(1+r_3\right)}$$

Recurrent relations are formulated as follows:

(SIV.13) $$\gamma_3^{(n+1)} = \gamma_3^{(n)} + h\cdot\left(r_3^{(n)} - r_1^{(n)}\right)$$

(SIV.14) $$m_3^{(n)} = m^{(n)}$$

We took value $h = 0.8$. Procedure of transformation based on recurrent relations (SII.12-14) leads to the exchange based on "production prices".

Table II(SIV). Exchange based on production prices.

Value of products consumed in each department (labor commanded):							
	C	V	M	m	SUM:	r	k
I	300.00	326.30	539.99	1.65	1166.30	0.86	1.09
II	250.10	1049.24	1035.35	0.99	2334.70	0.80	4.20
III	449.90	1125.45	1288.92	1.15	2864.27	0.82	2.50
SUM:	1000.00	2500.99	**2864.27**	1.15	**6365.26**	0.82	2.50

Matrix of input-flows in "production prices".							
	C	V	M	m	SUM:	r	k
I	360.07	300.18	539.99	1.799	**1200.25**	0.818	0.83
II	300.18	965.25	1035.35	1.073	**2300.79**	0.818	3.22
III	539.99	1035.35	1288.92	1.245	**2864.27**	0.818	1.92
SUM:	**1200.25**	**2300.79**	**2864.27**	1.245	**6365.30**	0.818	1.92
Value of product produced in departments (labor cost):							
	C	V	M	m	SUM:	r	k
I	300.00	326.30	373.70	1.15	1000.00	0.60	1.09
II	250.10	1049.24	1201.65	1.15	2500.99	0.92	4.20
III	449.90	1125.45	1288.92	1.15	**2864.27**	0.82	2.50
SUM:	1000.00	2500.99	**2864.27**	1.15	**6365.26**	0.82	2.50
gamma1	0.300	x	1.200	m1	1.655	m	1.145
gamma2	0.100	y	0.920	m2	0.987	r	0.818
gamma3	0.157	z	1.000	m3	1.145	k	2.501

The same color means the equal quantities.
Graphs bellow illustrate the dynamics of different economic quantities during the transition period.

Figure 1(SIV). Rates of surplus-value.

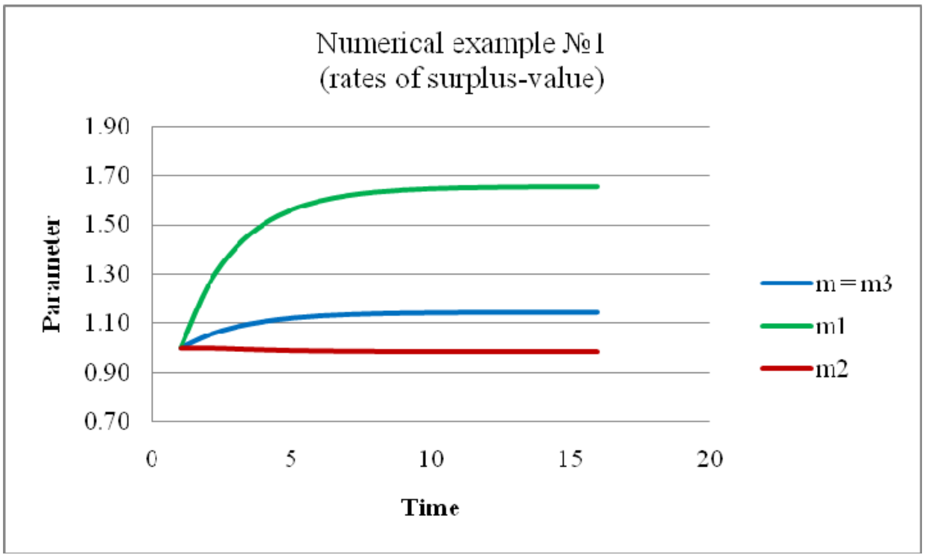


Figure 2(SIV). Technological coefficients $\gamma_1;\gamma_2;\gamma_3$.

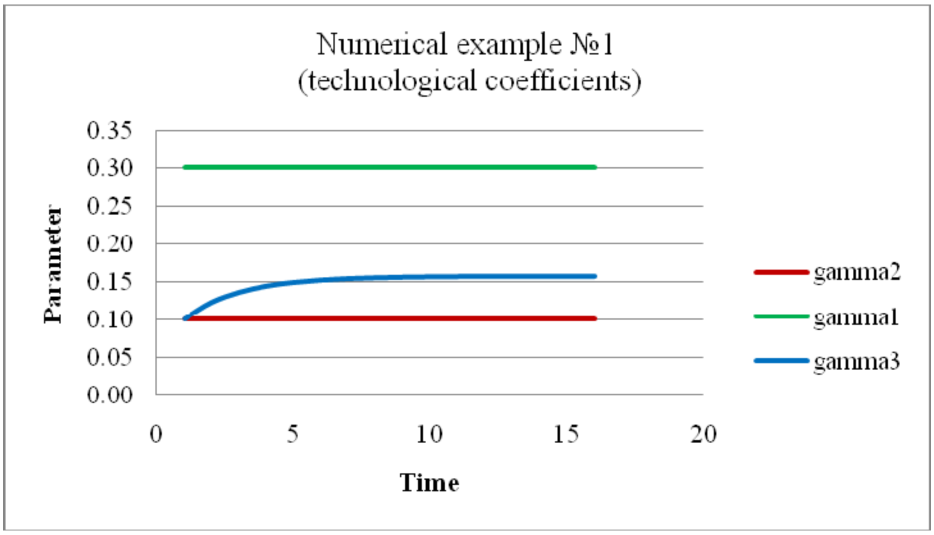


Figure 3(SIV). Rates of profit r_1; $r_2 = r_3$.

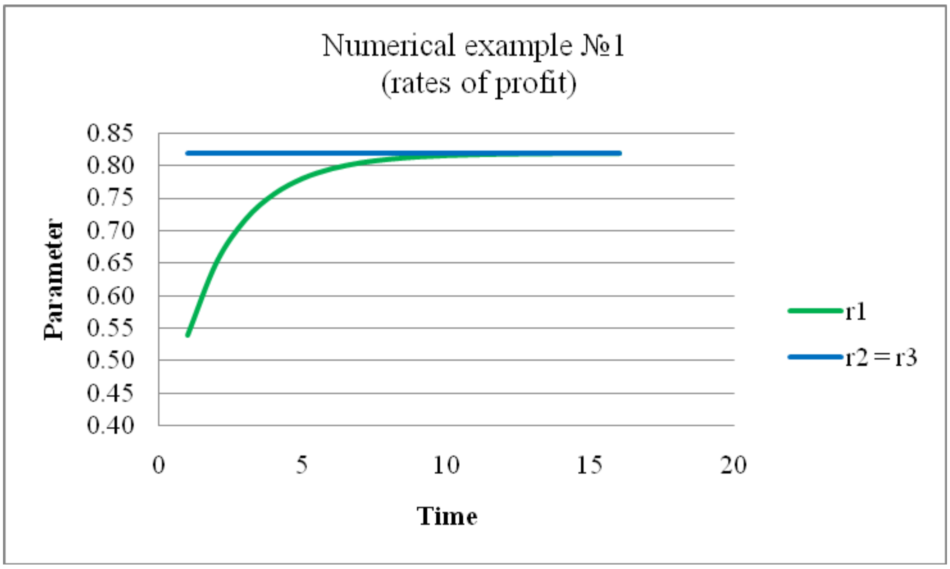


Figure 4(SIV). Temporary equilibrium prices $x; y;\ z = 1$.

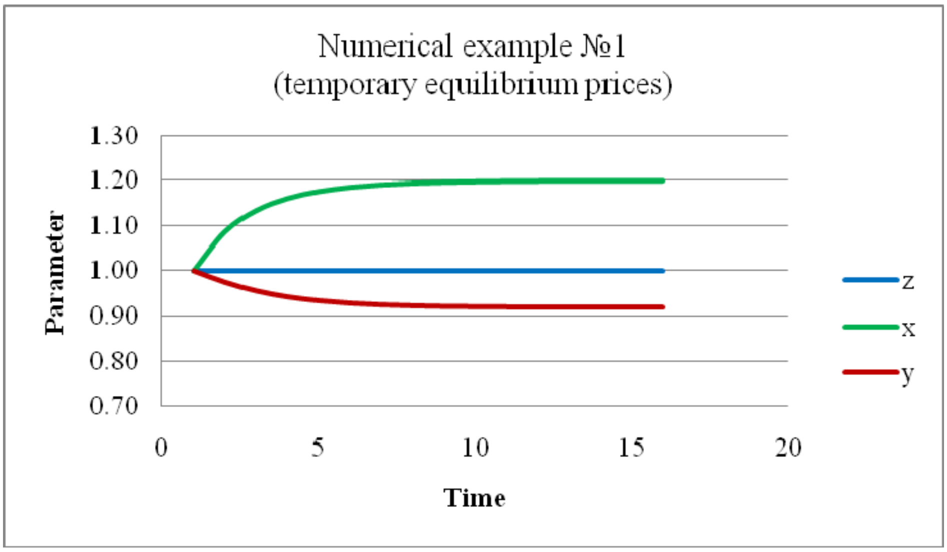


Figure 5(SIV). Organic compositions of capital (in values) $k_i = \dfrac{V_i}{C_i};\ i = \mathrm{I;II;III}$.

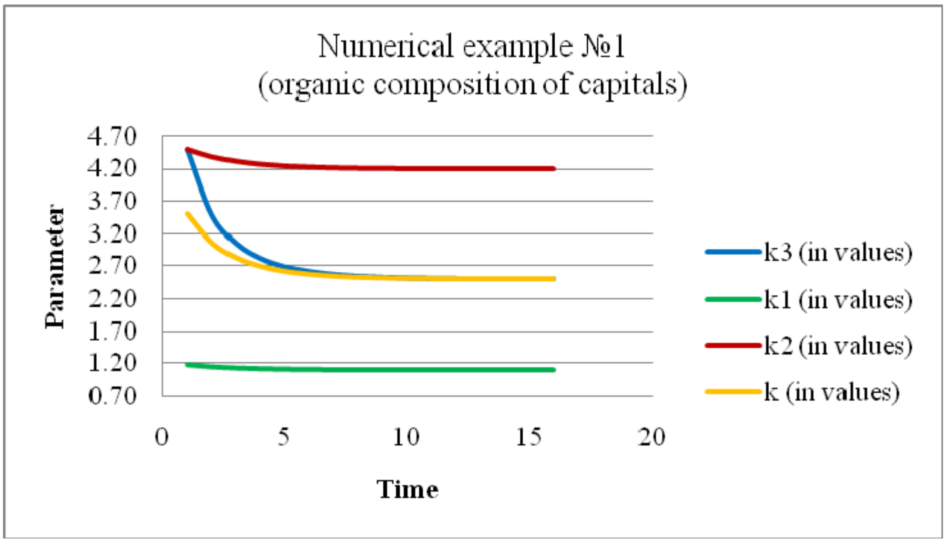
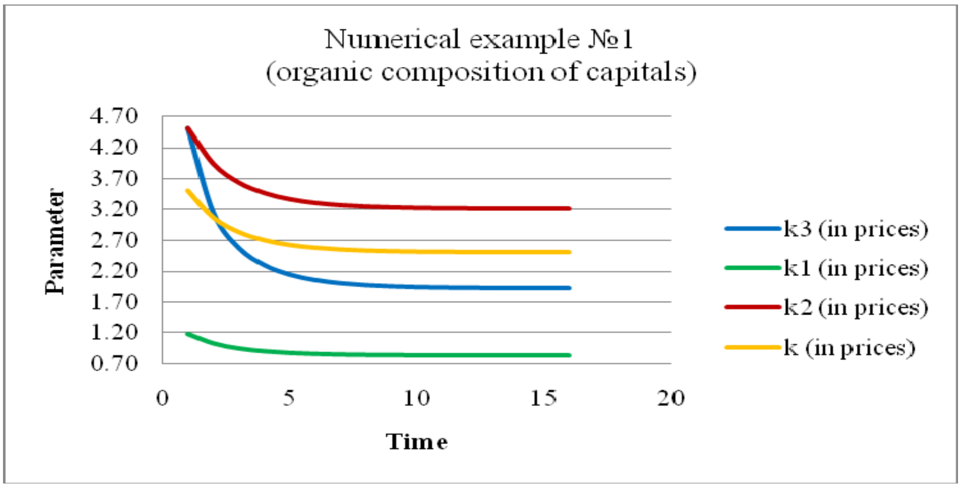
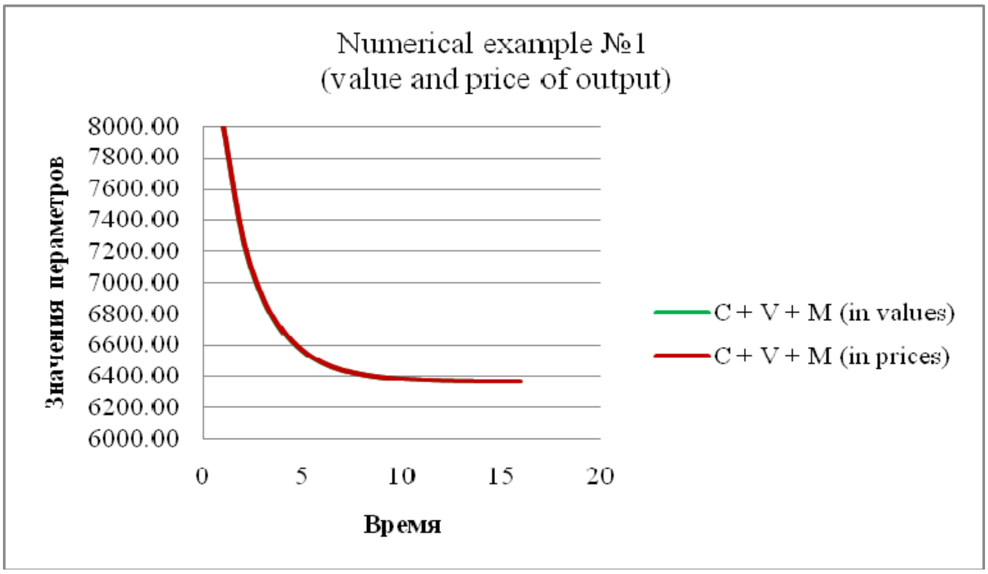


Figure 6(SIV). Organic compositions of capital (in temporary equilibrium prices).



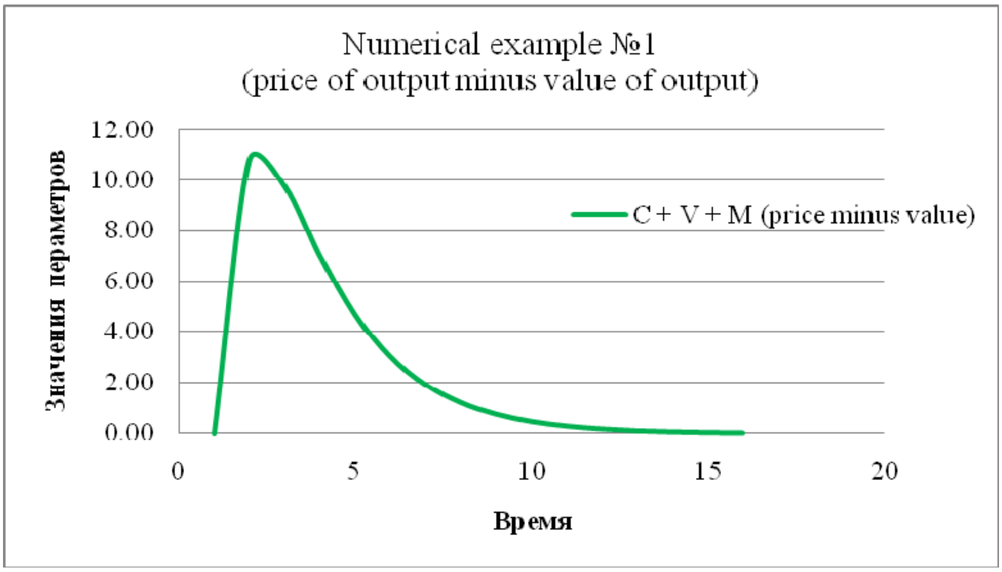
Marx's transformation rules are fulfilled to high precision in each sub-period of transition period. The rule (RI) (total surplus-value is equal to total profit) is fulfilled exactly whereas the rule (RII) (total value of output is equal to total temporary equilibrium price of output) is fulfilled to high precision.

Figure 7(SIV). Total value and price of output.



The difference between value and price of output is no more than 0.15% of output value. We see that graphs in Figure 7(SII) almost coincide. Graph of the difference is depictured in Figure 8(SII).

Figure 8(SIV). The difference between value and price of output.



Example №2. Parameter γ_3 decreases during transition period.

Let's take the following initial values of parameters $\gamma_1 = 0.3 < \gamma_2$; $\gamma_2 = \gamma_3 = 0.5$; $m = 1$.

Recurrent relations are the same as in the first example. We used value $h = 0.27$ for the second model.

Table III(SIV). Initial Matrix of Exchange based on "values".

Value of products consumed in each department (labor commanded):							
	C	V	M	m	Σ:	r	k
I	300	350	350	1.00	1000	0.54	1.17
II	350	175	175	1.00	700	0.33	0.50
III	350	175	175	1.00	700	0.33	0.50
Σ:	1000	700	700	1.00	2400	0.41	0.70

Table IV(SIV). Exchange based on production prices (after transformation).

Value of products consumed in each department (labor commanded):							
	C	V	M	m	Σ:	r	k
I	300.00	410.85	239.25	0.58	950.10	0.337	1.37
II	449.96	264.10	235.77	0.89	949.83	0.330	0.59
III	250.04	224.98	158.34	0.70	633.36	0.333	0.90
Σ:	1000.00	899.93	633.36	0.70	2533.29	0.333	0.90

Matrix of input-flows in "production prices".							
	C	V	M	m	Σ:	r	k
I	287.06	430.55	239.25	0.556	956.85	0.333	1.50
II	430.55	276.76	235.77	0.852	943.08	0.333	0.64
III	239.25	235.77	158.34	0.672	633.36	0.333	0.99
Σ:	956.85	943.08	633.36	0.672	2533.29	0.333	0.99
Value of product produced in departments (labor cost):							
	C	V	M	m	Σ:	r	k
I	300.00	410.85	289.15	0.70	1000.00	0.41	1.37
II	449.96	264.10	185.87	0.70	899.93	0.26	0.59
III	250.04	224.98	158.34	0.70	633.36	0.33	0.90
Σ:	1000.00	899.93	633.36	0.70	2533.29	0.33	0.90
gamma1	0.300	x	0.957	m1	0.582	m	0.704
gamma2	0.500	y	1.048	m2	0.893	r	0.333
gamma3	0.395	z	1.000	m3	0.704	k	0.900

The same color means the equal quantities.

Graphs bellow illustrates the dynamics of different economic values during the transition period.

Figure 9(SIV). Rates of surplus-value.

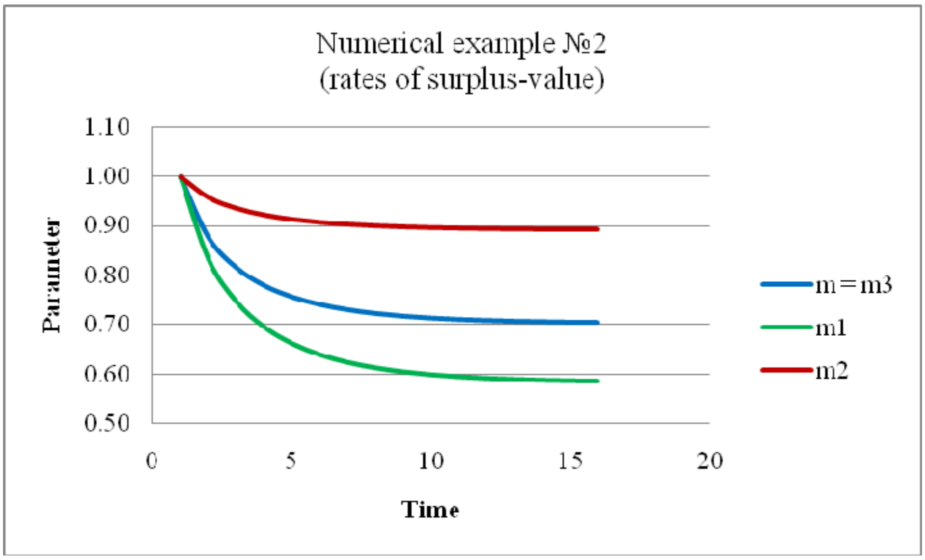


Figure 10(SIV). Technological coefficients $\gamma_1;\gamma_2;\gamma_3$.

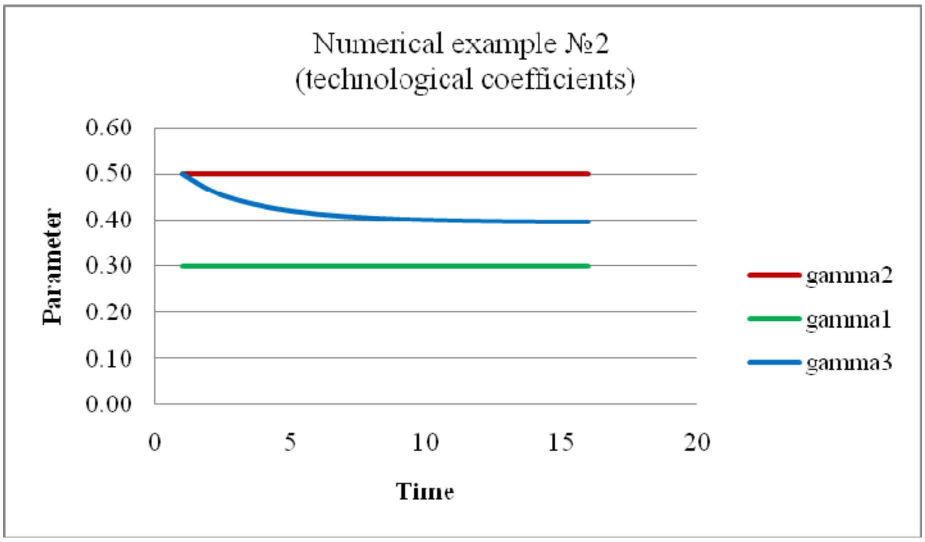


Figure 11(SIV). Rates of profit r_1; $r_2 = r_3$.

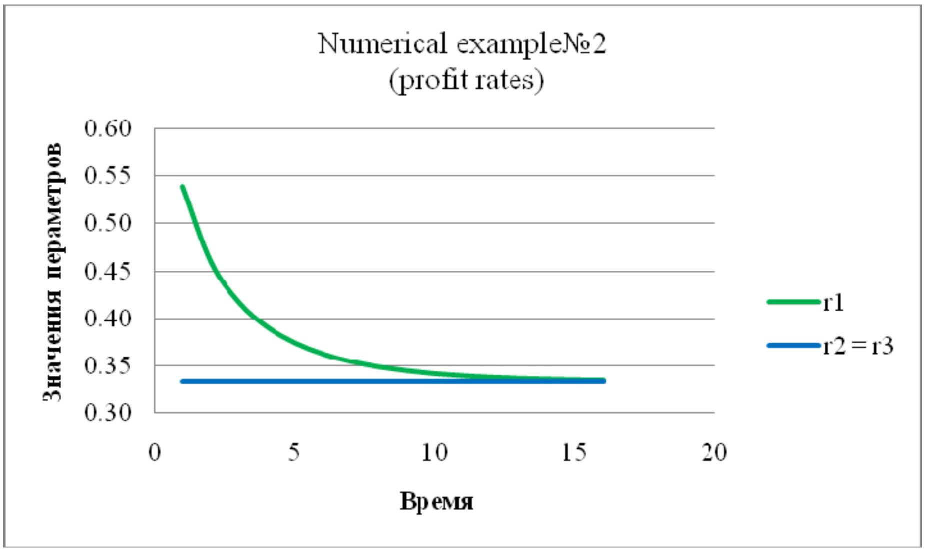


Figure 12(SIV). Temporary equilibrium prices $x; y; z = 1$.

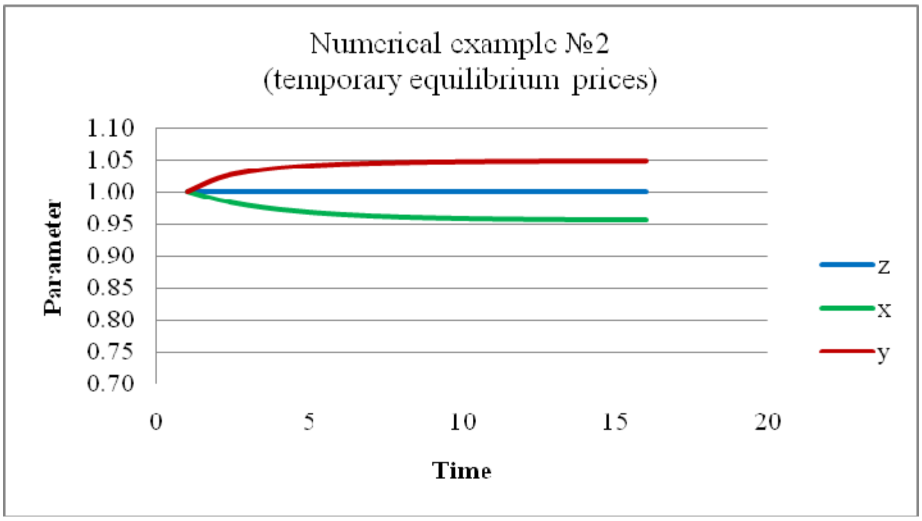


Figure 13(SIV). Organic compositions of capital (in values) $k_i = \frac{V_i}{C_i}$; $i = \mathrm{I;II;III}$.

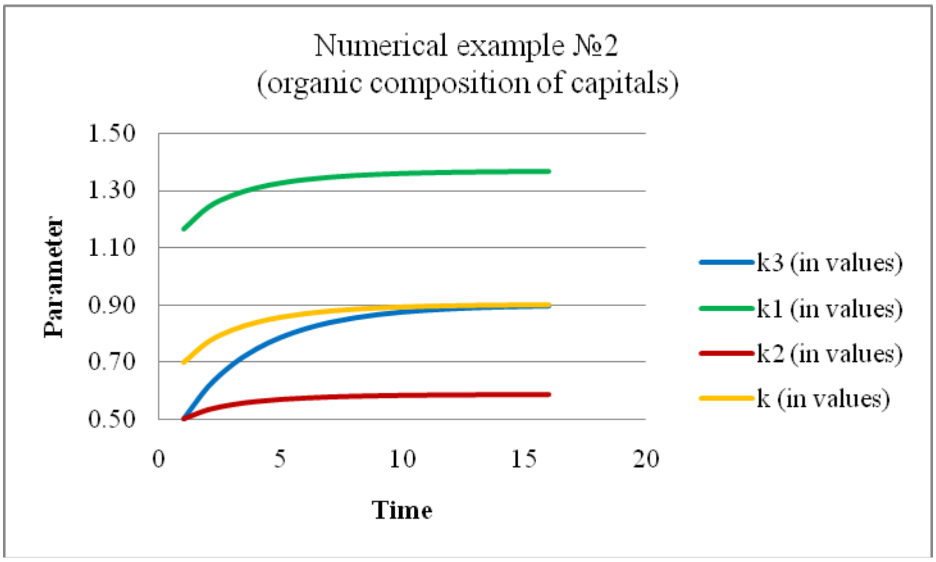


Figure 14(SIV). Organic compositions of capital (in temporary equilibrium prices).

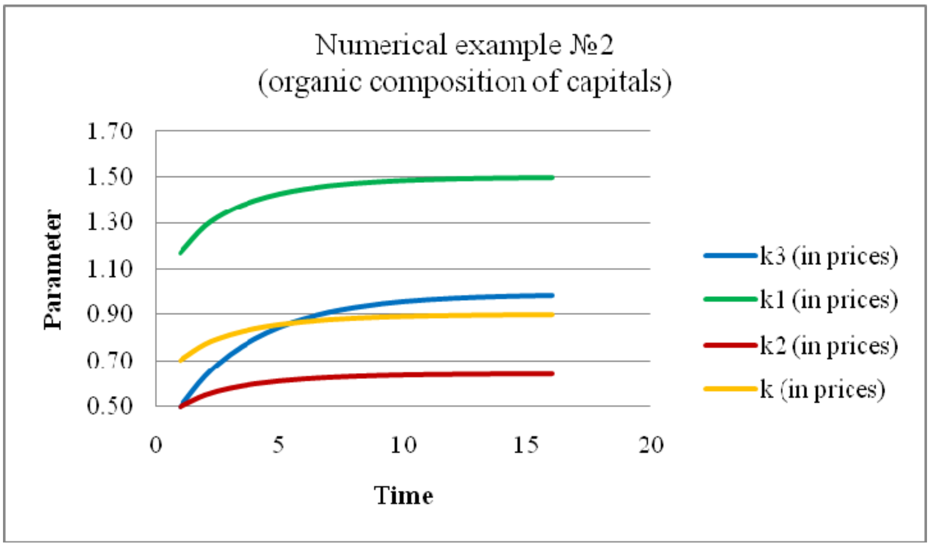


Figure 15(SIV). Total value and price of output.

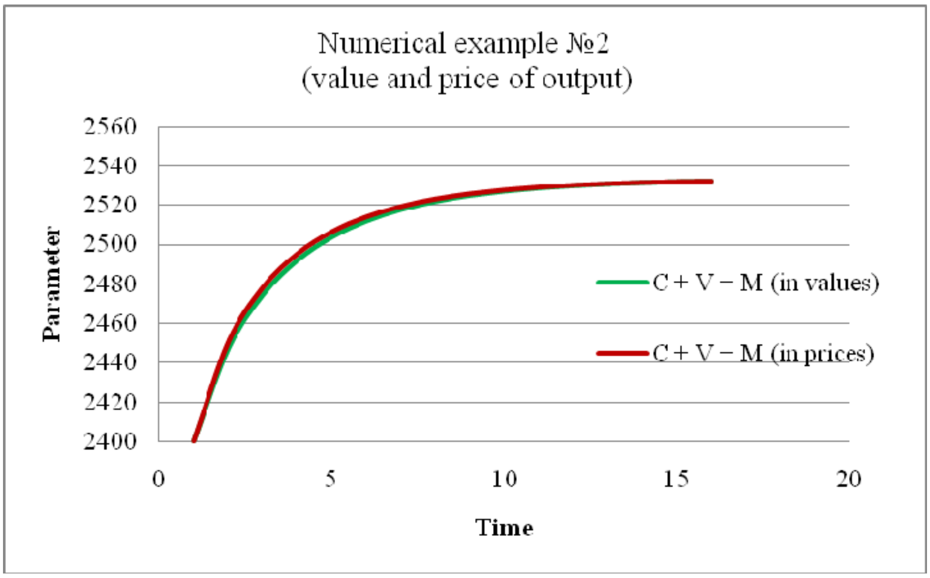
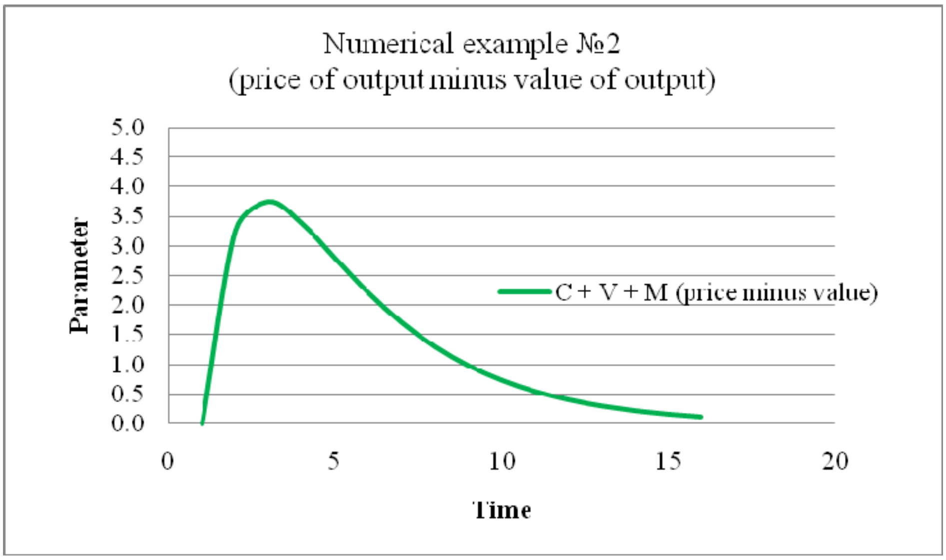


Figure 16(SIV). The difference between value and price of output.



Short discussion of results.

The qualitative properties of these models are presented in the following Table.

Table V. Qualitative properties of models (numerical examples №1 and №2).

	gamma3	m = m3	m1	m2	r1	k1	k2	k3	k	x	y	C+V+M
I	↑	↑	↑	↓	↑	↓	↓	↓	↓	↑	↓	↓
II	↓	↓	↓	↓	↓	↑	↑	↑	↑	↓	↑	↑

Symbol ↑ (↓) designate the increase (the decrease) of quantity during transition period. We see that properties of these solutions are antithetical. Transformation rules are fulfilled to high precision during transition period. Output (in values and in prices) decreases in the first case and it grows in the second case. Possibly jump in rate of growth during so-called “industrial revolution” can be explained partially by the influence of transformation.

SUPPLEMENT V. Numerical example of solution in the Model-2 (for asymmetric matrix of input-flows).

Value of products consumed in each department (labor commanded):						
	C (means of production)	V (consumer goods)	Mv (consumer goods)	Mm (luxury goods)	m	W
I	420.00	340.00	105.80	220.74	0.960	865.80
II	370.00	940.00	235.09	307.65	0.577	1545.09
III	210.00	225.17	127.47	65.73	0.858	562.64
SUM:	**1000.00**	1505.17	468.36	**594.12**	0.706	**3567.65**
	V + Mv =	**1973.53**	**M = Mv + Mm =**	**1062.47**		
	CAPITAL =	**2505.17**		**r = M' : (C' + V') =**		**0.4241**

Value of products produced in departments (labor cost):					
	C (transferring value)	V (necessary labor)	M (surplus labor)	m	W
I	420.00	340.00	240.00	0.706	**1000.00**
II	370.00	940.00	663.53	0.706	**1973.53**
III	210.00	225.17	158.95	0.706	**594.12**
SUM:	1000.00	1505.17	**1062.47**	0.706	**3567.65**
	CAPITAL =	**2505.17**		**r = M' : (C' + V') =**	**0.4241**

Prices of production:						
	C (means of production)	V (consumer goods)	Mv (consumer goods)	Mm (luxury goods)	r	W
I	467.79	314.30	97.80	233.89	0.424	**1113.78**
II	412.10	868.94	217.32	325.98	0.424	**1824.35**
III	233.89	208.15	117.83	69.64	0.424	**629.52**
SUM:	**1113.78**	1391.39	432.95	**629.52**	0.424	**3567.65**
	CAPITAL =	**2505.17**		**r = P : (C + V) =**		**0.4241**
	V + Mv =	**1824.35**	**P = Mv + Mm =**	**1062.47**		
	x	**1.114**	**alpha1**	**0.295**	**k**	**1.505**
	y	**0.924**	**alpha2**	**0.400**		
	z	**1.060**	**alpha3**	**0.629**		
	m	**0.706**	**r**	**0.424**		

SUPPLEMENT VI. The proof of execution of non-trivial balance-conditions in Model-4 with the next equalities between parameters: $\alpha_1 = \alpha$ and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$.

The following **Program of computations in "Mathematica 5.2"** was written.

```
Solve[
 {(C1 + C2 + C3) C2 t^2 +
    (V2 (1 - g) (C1 + C2 +C3) -
       C1 (V1 + V2 + V3)
         (1 + ((C2 + C3 - V1) / V1)
             ((V1 + V2 + V3) (C2 +C3 - V1) - C3 V1 -
                V3 (C2 + C3)) /
              ((V1 + V2 + V3) (C2 +C3 - V1))))) t -
    (V1 + V2 + V3) V1 (1 -g)
     (1 + ((C2 + C3 - V1) / V1)
         ((V1 + V2 + V3) (C2 +C3 - V1) - C3 V1 - V3 (C2 + C3)) /
          ((V1 + V2 + V3) (C2 +C3 - V1))) == 0,
  g - 1 +
    ((C2 t + V2 (1 - g))
         (C1 + C2 + C3 + (V1 + V2 + V3) (C2 + C3) (1 / V1)) -
       (C1 + C2 + C3) (V1 + V2 + V3)
         (1 + ((C2 + C3 - V1) / V1)
             ((V1 + V2 + V3) (C2 +C3 - V1) - C3 V1 -
                V3 (C2 + C3)) /
              ((V1 + V2 + V3) (C2 +C3 - V1)))) /
     ((V1 + V2 + V3) ^2
        (1 + ((C2 + C3 - V1) / V1)
            ((V1 + V2 + V3) (C2 +C3 - V1) - C3 V1 -
               V3 (C2 + C3)) /
             ((V1 + V2 + V3) (C2 +C3 - V1)))) == 0,
  C2 t -
     V1
      (1 + ((C2 + C3 - V1) / V1)
          ((V1 + V2 + V3) (C2 +C3 - V1) - C3 V1 - V3 (C2 + C3)) /
           ((V1 + V2 + V3) (C2 +C3 - V1))) - h == 0},
 {t, g, h}]
```

We used the following designations: $\gamma \to g$, $t = \dfrac{x}{y}$, $h = t \cdot C_2 - V_1\left[1 + \left(1 - \alpha\right) m\right]$. Non-trivial balance-conditions are carried out if $h = 0$.

This Program solves the following system of equations:

$$\begin{cases} CC_2t^2 + \left[CV_2(1-\gamma) - VC_1\left(1+m(1-\alpha)\right)\right]t - VV_1(1-\gamma)\left(1+m(1-\alpha)\right) = 0; \text{ since } \gamma_1 = \gamma_2 = \gamma & \text{(E1)} \\ \gamma = 1 - \dfrac{\left[C_2t + V_2(1-\gamma)\right]\cdot\left[C + V(1+m)\right] - CV\left[1+m(1-\alpha)\right]}{\left[1+m(1-\alpha)\right]V^2} & \text{(E2)} \\ m = \dfrac{C_2 + C_3}{V_1} - 1 & \text{(E3)} \\ \alpha = \dfrac{C_3 + V_3(1+m)}{mV} & \text{(E4)} \\ h = t\cdot C_2 - V_1\left[1+(1-\alpha)m\right]; \text{ since } \alpha_1 = \alpha & \text{(E5)} \end{cases}$$

Parameters m and α are excluded from equations of Program by means of relations (E3) – (E5).

SOLUTION OF SYSTEM:

$$\Big\{\Big\{h \to 0,$$

$$g \to \Big(C1\,C2 + C2^2 + C2\,C3 + C2\,V1 - C1\,V2 - C3\,V2 + \frac{C1\,C2\,V2}{V1} + \frac{C2^2\,V2}{V1} + \frac{C1\,C3\,V2}{V1} + \frac{2\,C2\,C3\,V2}{V1} + \frac{C3^2\,V2}{V1} + C2\,V3 - \frac{C1\,(C2\,V1 + C2\,V2 + C3\,V2)}{V1 + V2 + V3} - \frac{2\,C2\,(C2\,V1 + C2\,V2 + C3\,V2)}{V1 + V2 + V3} - \frac{2\,C3\,(C2\,V1 + C2\,V2 + C3\,V2)}{V1 + V2 + V3} - \frac{C2\,V2\,(C2\,V1 + C2\,V2 + C3\,V2)}{V1\,(V1 + V2 + V3)} - \frac{C3\,V2\,(C2\,V1 + C2\,V2 + C3\,V2)}{V1\,(V1 + V2 + V3)} - \frac{C2\,(C2\,V1 + C2\,V2 + C3\,V2)\,V3}{V1\,(V1 + V2 + V3)} - \frac{C3\,(C2\,V1 + C2\,V2 + C3\,V2)\,V3}{V1\,(V1 + V2 + V3)}\Big) \Big/ (C2\,V1 - C1\,V2 - C3\,V2 + C2\,V3),$$

$$t \to \frac{C2\,V1 + C2\,V2 + C3\,V2}{C2\,(V1 + V2 + V3)}\Big\},$$

$$\Big\{h \to \frac{-C3\,V2 + C2\,V3}{V1 + V2 + V3},\quad g \to \frac{-C2 - C3 + V1}{V1},\quad t \to 1\Big\}\Big\}$$

We see that $h = 0$ in the first solution and therefore non-trivial balance-conditions are carried out. The second solution is not realistic since in this case $\gamma = -m < 0$.

SUPPLEMENT VII. Numerical example of non-realistic solution in the Model-4.

	Value of products consumed in each department (labor commanded)								
	C	V work.	V cap.	V	Mv	Mm	M	m	W''
I	2000	410	90	500	165	2472	2637	5.27	5137
II	1000	320	80	400	53	1334	1386	3.47	2786
III	2800	-1177	1277	100	2822	-246	2576	25.76	5476
SUM:	**5800**	-447	1447	1000	3040	**3560**	6600		**13400**
	CAPITAL (K) =	**5353**							
r = (M + V cap.) : (C + V work.) =			**1.503**	V + Mv =	**4040**		P = M + V cap. =		**8047**

MODEL - 4. Non-trivial balance-conditions may be violated.

	Value of products produced in departments (labor costs)							
	C	V work.	V cap.	V	M	m	W	
I	2000	410	90	500	3300	6.60	**5800**	W1' = W 1*x
II	1000	320	80	400	2640	6.60	**4040**	W2' = W2 *y
III	2800	-1177	1277	100	660	6.60	**3560**	W3'=W3*z
SUM:	**5800**	-447	1447	1000	6600	6.60	**13400**	
	CAPITAL (K) =	**5353**			P = M + V cap. =		**8047**	

	Prices of production								
	C'	V work.	V cap.	V'	M'v	M'm	P' = r * K	r'	W'
I	1963	311	68	379	125	3225	3418	1.503	**5692**
II	981	243	61	303	40	1740	1840	1.503	**3064**
III	2748	-893	969	76	2141	-321	2789	1.503	**4644**
SUM:	**5692**	-339	1098	759	2306	**4644**	**8047**	**1.503**	**13400**
	CAPITAL (K) =	**5353**		V' + M'v =	**3064.37**			alpha =	0.539
	gamma 1 =	0.180		gamma2 =	0.200	gamma3 =	12.772	gamma =	1.447
	Cn' = Cn * x				Kn = Cn + V раб.			x =	0.981
	V'n раб. = Vn раб. * y				K'n = C'n + V' раб.			y =	0.759
	V'n кап. = Vn кап. * y				P'n = r * K'n			z =	1.304
	V' = V * y				W'n = (1 + r) * K'n			alpha1 =	0.950
	M'vn = Mvn * y				V'n кап. + M'vn + M'mn = P'n			alpha2 =	0.980
	M'mn = Mmn * z				r' = P' : K'			alpha3 =	-3.276

Exogenous coefficients of value-matrix are marked by green color.

SUPPLEMENT VIII.

Technical Excel-file contain programs of calculations.
Worksheet “SV” = (SV)-structure;
Worksheet “SP” = (SP)-structure;
Worksheet “SP-SV” = structure when exchange based on “values” coincides with exchange based on “production prices”;
Worksheet “Ex1” = model of “historical transformation” (example №1);
Worksheet “Ex2” = model of “historical transformation” (example №2);
Worksheet “Mod-2” = numerical example of solution in the Model-2;
Worksheet “Mod-2(Marx)” = Marx’s numerical example for the Model-2;
Worksheet “Mod-2-2” = NTBC-invariant conversion of Model-2 into a new Model-2 for Marx’s numerical example;
Worksheet “Mod-2-1” = NTBC-invariant conversion of Model-2 into Model-1 for Marx’s numerical example;
Worksheet “Mod’-2-2” = NTBC-invariant conversion of Model-2 into a new Model-2;
Worksheet “Mod’-2-1” = NTBC-invariant conversion of Model-2 into Model-1;
Worksheet “Mod’-4” = Model-4
Worksheet “Mod-4” = Model-4 satisfying to NTBC and NTBC-invariant conversation into Model-2;
Worksheet “Mod-4-2-1” = NTBC-invariant conversation Model-4 → Model-2 → Model-1;
Worksheet “Mod-4-nonrealistic” = numerical example of nonrealistic solution in Model-4;
Worksheets “RandCap1”; “RandCap2” and “RandCap3” = program for Model-4 with random distribution of capitals.